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The integrability of a class of cubic Lotka–Volterra systems

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ABSTRACT

The integrability of a class of cubic Lotka–Volterra systems $\dot{x} = x(1 - a_0x^2 - a_1xy - a_2y^2)$, $\dot{y} = y(-\lambda + b_0x^2 + b_1xy + b_2y^2)$, is studied. For odd λ satisfying $\lambda \geq 3$, we derive the necessary and sufficient conditions for the integrabilities of the above systems.

1. Introduction

Consider planar vector fields with $p : -q$ resonant singular point

$$\dot{x} = px + P(x, y), \qquad \dot{y} = -qy + Q(x, y),$$

where $p, q \in N$ and P, Q are polynomials. The classic center problem by Dulac [1] is to find the conditions for the existence of a local analytic first integral $H(x, y) = x^q y^p + \cdots$ for the above systems.

For the $1 : -1$ resonant case, the above problem is completely solved when P and Q are homogeneous polynomials of degrees 2 (by Dulac), and 3 (by Sibirskii). For the $1 : -2$ resonant case, the above problem is almost certainly solved in [2,3] when P and Q are homogeneous polynomials of degrees 2. For more results, we recommend the readers the papers [4–6], etc.

Here we are interested in the Lotka–Volterra type systems

$$\dot{x} = x(p + P(x, y)), \qquad \dot{y} = y(-q + Q(x, y)). \tag{1}$$

The form of L–V systems seems to be simple, but even in the simplest case $\deg P = \deg Q = 1$, the family of Lotka–Volterra is sufficiently rich to exhibit most general features that are expected from the general polynomial families: existence of non-normalizable systems, existence of integrable and nonlinearizable systems, etc.

The integrability of classic L–V systems, i.e. P and Q are homogeneous polynomials of degree 1, has been studied by many authors. In [2,7], all the integrable conditions are given for $1 : -n$ or $2 : -n$ resonant cases, where $n \in \mathbb{N}$. For generic $p : -q$ resonant cases, some new integrable sufficient conditions are given in [8,9]. For small (p, q) $(p + q \leq 12)$, in [8] the above sufficient conditions are shown to be also necessary.

Some authors also consider the integrability of L–V systems with homogeneous polynomials P, Q of degree 2 in [5,10–12], degree 3 in [13] and degree 4 in [14]. When P and Q are quadratic homogeneous polynomials, system (1) can

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be written as

$$\dot{x} = x(1 - a_0x^2 - a_1xy - a_2y^2) \qquad \dot{y} = y(-\lambda + b_0x^2 + b_1xy + b_2y^2), \tag{2}$$

where $\lambda = \frac{q}{p}$.

The case $\lambda = 3$ is studied in [5] where in fact the general homogeneous cubic $1 : -3$ resonant case was treated. In [12], the authors consider the case $\lambda = \frac{3}{2}$. For $\lambda \in \mathbb{N}$, some sufficient conditions are given in [10,11], and for small odd λ ($\lambda \leq 9$), the authors of [10,11] obtain necessary and sufficient conditions for the integrabilities of system (2). In their proofs, they need to use mathematic tools, such as Maple, to calculate the first three saddle quantities of system (2). For small λ, this can be easily realized by computers, but if λ is very large, for example $\lambda > 10^{100}$, then it is difficult for computers even to factorize λ as the product of prime numbers; of course it is impossible for computers to calculate the saddle quantities. Thus the methods in [11,10] cannot work.

In this paper, for odd λ, we do not use computers, but use theoretical analysis to obtain the first three saddle quantities of system (2). In fact, the saddle quantities of system (2) that we obtain are the same as that in [10,11], but our method works not only for small λ, but also for all the odd λ. Concretely, our result is as follows:

Theorem 1. *For odd λ satisfying $\lambda \geq 3$, system (2) is integrable if and only if one of the following conditions is satisfied:*

(a) $\displaystyle\prod_{i=0}^{(\lambda-5)/2} (b_0 - (2i+1)a_0) = 0;$

(b) $f_1 = 0, \qquad b_0 = (\lambda - 2)a_0;$

(c) $\displaystyle f_1 = 0, \qquad a_1 = 0, \qquad \prod_{i=0}^{(\lambda-3)/2} (b_0 - 2ia_0) = 0;$

(d) $f_1 = 0, \qquad f_2 = 0,$

where

$$f_1 = \lambda a_0a_1 + a_1b_0 + (\lambda - 2)a_0b_1 - b_0b_1;$$
$$f_2 = \lambda a_0a_2 + (\lambda - 1)a_0b_2 - b_0b_2.$$

Of course, the condition (a) here exists only for $\lambda \geq 5$.

Unfortunately, when λ is even, the saddle quantities of system (2) will be much more complicated, so Lemma 3 in this paper is not right, and hence our method does not work.

2. The sufficient conditions

For $\lambda \in \mathbb{N}$, all the sufficient conditions here have been obtained in [11,10]; here we will extend their results to the case $\lambda \in \mathbb{Q}^+$. Since our proof is almost the same as their, we only give the outline of the proof.

Lemma 2. *If $\lambda \in \mathbb{Q}^+$, then system (2) is integrable if one of the following conditions is satisfied:*

(a) $\displaystyle\prod_{0\leq i\leq(\lambda-5)/2} (b_0 - (2i+1)a_0) = 0;$

(b) $f_1 = 0, \qquad b_0 = (\lambda - 2)a_0;$

(c) $\displaystyle f_1 = 0, \qquad a_1 = 0, \qquad \prod_{i=0}^{(\lambda-3)/2} (b_0 - 2ia_0) = 0;$

(d) $f_1 = 0, \qquad f_2 = 0.$

Proof. The conditions (a)–(c) are equivalent to the following three conditions

$$b_0 = (2m+1)a_0, \quad 0 \leq m \leq \frac{q-5}{2};$$
$$a_1 = 0, \qquad b_0 = (\lambda - 2)a_0;$$
$$a_1 = b_1 = 0, \qquad b_0 = 2ma_0, \quad 0 \leq m \leq \frac{q-3}{2},$$