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Attractors for Navier–Stokes flows with multivalued and nonmonotone subdifferential boundary conditions

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This paper is dedicated to the memory of Zdzisław Naniewicz

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ABSTRACT

We consider two-dimensional nonstationary Navier–Stokes shear flow with multivalued and nonmonotone boundary conditions on a part of the boundary of the flow domain. We prove the existence of global in time solutions of the considered problem which is governed by a partial differential inclusion with a multivalued term in the form of Clarke subdifferential. Then we prove the existence of a trajectory attractor and a weak global attractor for the associated multivalued semiflow.

This research is motivated by control problems for fluid flows in domains with semipermeable walls and membranes.

1. Introduction

In this paper we consider two-dimensional nonstationary incompressible Navier–Stokes shear flows with nonmonotone boundary conditions on a part of the boundary of the flow domain. Our aim is to prove the existence of global in time solutions of the considered problem which is governed by a partial differential inclusion, and then to prove the existence of a trajectory attractor and a weak global attractor for the associated multivalued semiflow.

This research is motivated by control problems for fluid flows in domains with semipermeable walls and membranes.

The problem we consider is as follows. The flow of an incompressible fluid in a two-dimensional domain Ω is described by the equation of motion

$$u_t - \nu \Delta u + (u \cdot \nabla)u + \nabla p = 0 \quad \text{for } (x,t) \in \Omega \times \mathbb{R}^+, \tag{1.1}$$

where the viscosity coefficient $\nu > 0$, and the incompressibility condition

$$\operatorname{div} u = 0 \quad \text{for } (x,t) \in \Omega \times \mathbb{R}^+. \tag{1.2}$$

To define the domain Ω of the flow let us consider the channel

$$\Omega_\infty = \{x = (x_1, x_2) : -\infty < x_1 < \infty,\ 0 < x_2 < h(x_1)\},$$

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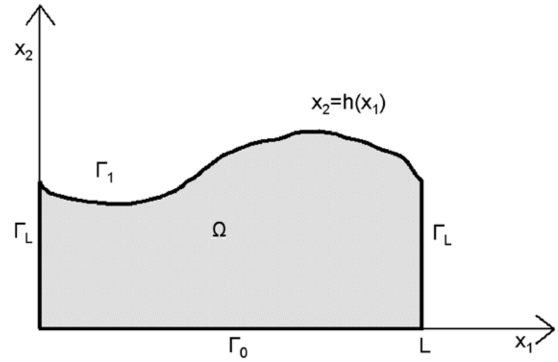

Fig. 1. Schematical view of Ω.

where the function $h:\mathbb{R}\to\mathbb{R}$ is a positive, smooth, and L-periodic. Then we set

$$\Omega=\{x=(x_1,x_2):0<x_1<L,\ 0<x_2<h(x_1)\}$$

and $\partial\Omega=\bar{\Gamma}_0\cup\bar{\Gamma}_L\cup\bar{\Gamma}_1$, where Γ_0 and Γ_1 are the bottom and the top, and Γ_L is the lateral part of the boundary of Ω. The domain Ω is schematically presented in Fig. 1.

We are interested in solutions of (1.1)–(1.2) in $\Omega\times\mathbb{R}^+$ which are L-periodic with respect to x_1. We assume that

$$u=0\quad \text{at } \Gamma_1\times\mathbb{R}^+. \tag{1.3}$$

On the bottom Γ_0 we impose the following conditions. The tangential component u_T of the velocity vector on Γ_0 is given, namely, for some $s\in\mathbb{R}$,

$$u_T=u-u_N n=(s,0)\quad \text{at } \Gamma_0\times\mathbb{R}^+,\ \text{where } u_N=u\cdot n. \tag{1.4}$$

Furthermore, we assume the following subdifferential boundary condition

$$\tilde{p}(x,t)\in\partial j(u_N(x,t))\quad \text{at } \Gamma_0\times\mathbb{R}^+, \tag{1.5}$$

where $\tilde{p}=p+\frac{1}{2}|u|^2$ is the total pressure (called the Bernoulli pressure), $j:\mathbb{R}\to\mathbb{R}$ is a given locally Lipschitz superpotential, and ∂j is a Clarke subdifferential of j. For a locally Lipschitz function $j:\mathbb{R}\to\mathbb{R}$ its Clarke subdifferential $\partial j:\mathbb{R}\to 2^{\mathbb{R}}$ is defined as

$$\partial j(s)=conv\{\lim_{n\to\infty}j'(s_n)\ :\ s_n\to s, s_n\notin S\cup N_j, j'(s_n)\text{ converges}\},$$

where N_j is the null measure set on which j fails to be differentiable and S is any null measure set. Moreover the generalized directional derivative in the sense of Clarke of the function j at the point u in the direction r is defined as $j^0(u;r)=\max_{\xi\in\partial j(u)}\xi r$ (see for example [1,2] for the properties of Clarke subdifferential).

Let, moreover,

$$u(0)=u_0\quad \text{in } \Omega. \tag{1.6}$$

The considered problem is motivated by the examination of a certain two-dimensional flow in an infinite (rectified) journal bearing $\Omega\times(-\infty,+\infty)$, where $\Gamma_1\times(-\infty,+\infty)$ represents the outer cylinder, and $\Gamma_0\times(-\infty,+\infty)$ represents the inner, rotating cylinder. In the lubrication problems the gap h between cylinders is never constant. We can assume that the rectification does not change the equations as the gap between cylinders is very small with respect to their radii.

A physical interpretation of the boundary condition (1.5) can be as follows. The superpotential j in our control problem is not convex as it corresponds to the *nonmonotone* relation between the normal velocity u_N and the total pressure $\tilde{p}$ at Γ_0. Assuming that, left uncontrolled, the total pressure at Γ_0 would increase with the increase of the normal velocity of the fluid at Γ_0, we control $\tilde{p}$ by a hydraulic device which opens wider the boundary orifices at Γ_0 when u_N attains a certain value and thus $\tilde{p}$ *drops* at this value of u_N. Particular examples of such relations are provided in [3,4].

The knowledge or the judicious choice of the boundary conditions on the fluid–solid interface is of particular interest in lubrication area which is concerned with thin film flow behaviour. The boundary conditions to be employed are determined by numerous physical parameters characterizing, for example, surface roughness and rheological properties of the fluid.

The system of Eqs. (1.1)–(1.2) with *non-slip* boundary conditions: (1.3) at Γ_1 for $h=const$ and $u=const$ on Γ_0 (instead of (1.4)–(1.5) on Γ_0) was intensively studied in several contexts, some of them mentioned in the introduction of [5]. The autonomous case with $h\neq const$ and with $u=const$ on Γ_0 was considered in [6,7] and the nonautonomous case $h\neq const$, $u=U(t)e_1$ on Γ_0 was considered in [8]. Existence of exponential attractors for the Navier–Stokes fluids and global attractors for Bingham fluids, with the Tresca boundary condition on Γ_0 was proved in [9,10], respectively. Recently, attractors for two dimensional Navier–Stokes flows with Dirichlet boundary conditions were studied in [11], where the time continuous