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Traveling waves in porous media with phase-dependent damping

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ABSTRACT

We consider a simple model for the fluid flow in a porous medium. The model consists of a hyperbolic system of balance laws, which take into account phase changes and allow for metastable states thanks to the introduction of an equilibrium pressure. A damping term is included as well, which depend not only on the velocity but also on the phase of the fluid; in particular, it vanishes in the vapor phase. The existence and uniqueness of traveling waves is proved in several important cases.

1. Introduction

In this paper we consider a model for the isothermal and inviscid fluid flow through a porous medium, in presence of liquid–vapor phase changes. The system of evolution equations governing such flows in Lagrangian coordinates is

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v,\lambda)_x = -\alpha(\lambda)u, \\ \lambda_t = \dfrac{1}{\tau}\,(p(v,\lambda) - p_e)\,\lambda(\lambda-1), \end{cases} \tag{1.1}$$

for $t > 0$ and $x \in \mathbb{R}$. Here above, $v > 0$ denotes the specific volume, u the velocity and $\lambda \in [0,1]$ the mass–density fraction of the vapor in the fluid. The pressure function $p(v,\lambda) > 0$ is assumed to be of class C^2 and satisfies

$$p_v < 0, \qquad p_\lambda > 0, \qquad p_{vv} > 0, \qquad p_{v\lambda} < 0. \tag{1.2}$$

We observe that by (1.2) it easily follows that $\lim_{v\to+\infty} p_v(v,\lambda) = 0$. Physical pressure functions satisfy, in addition to (1.2), also

$$\lim_{v\to 0+} p(v,\lambda) = +\infty, \qquad \lim_{v\to+\infty} p(v,\lambda) = 0, \tag{1.3}$$

for every $\lambda \in [0,1]$. In turn, the first condition implies $\lim_{v\to 0+} p_v(v,\lambda) = -\infty$. The pressure function $p = (\kappa_0 + \lambda(\kappa_1 - \kappa_0))v^{-\gamma}$ satisfies both (1.2) and (1.3) for $\gamma \geq 1$ and $\kappa_0 < \kappa_1$. The pressure law $p_s = p_e + c^2\{1/v - [\lambda/v_g + (1-\lambda)/v_l]\}$, where $v_g > v_l$ are reference values for the specific volumes of gas and liquid, can be deduced by the stiffened gas model; it satisfies the conditions in (1.2) apart from the last one: $(p_s)_{v\lambda} \equiv 0$. We always assume (1.2) while the further assumption (1.3) is only required is some special cases.

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The assumption $p_v < 0$ in (1.2) implies that (1.1) is a hyperbolic system of balance laws with eigenvalues $\pm\sqrt{-p_v}$ and 0. Above, we denoted by $p_e > 0$ a (constant) equilibrium pressure and by $\tau > 0$ a characteristic reaction time.

The flow is hosted in a medium that induces a friction force $-\alpha u$ proportional to the linear momentum, with $\alpha \geq 0$. In general, the friction coefficient α depends on the porosity of the material: the higher the porosity, the lower is α, since then it is easier for the fluid to flow through. Moreover, α also depends on the fluid and in particular on the phase of the fluid: the medium resists the flow of vapor much less than it does to the flow of liquid. In this paper, we assume for simplicity that the material is uniform, so that α is independent of the porosity, but allow α to depend on λ. This framework includes the interesting case when α is strictly positive in the liquid phase and vanishes in the vapor phase. For different approaches to liquid/vapor flows in porous media we quote the Baer–Nunziato model proposed in [1] and the Stefan-like model in [2]. Both models retain more physical information on the phenomenon than ours but can be fully treated only in a numerical way. Our model is much simpler but allow for a much more detailed mathematical analysis.

We now briefly discuss some known results on (1.1). If the third equation is missing and then p only depends on v, the related system has been studied by many authors, see for instance [3–7]. If both source terms are missing in (1.1), the resulting homogeneous system has been studied in [8,9]; see also [10] for a large data analysis. About the complete system (1.1), the case $\alpha \equiv 0$ has been considered in [11] with a special emphasis to relaxation approximation (see also [12] for the limit case $\tau = \infty$); the study of the traveling waves was done in [13]. The case when α is a positive constant was studied in [14]; entirely analogous results are valid if α depends on λ but is bounded away from zero [15]. Moreover, in [15] we proved that, under such an assumption, system (1.1) satisfies the Shizuta–Kawashima condition and is strictly entropy-dissipative; as a consequence [16], the initial-value problem has smooth global solutions if the initial data are close to the stable-liquid phase ($\lambda = 0$ and $p > p_e$) or to the stable-vapor phase ($\lambda = 1$ and $p < p_e$).

In this paper we investigate the existence of traveling waves for system (1.1) in the case $\alpha(\lambda)$ is allowed to vanish: more precisely, we assume that α is smooth and satisfies

$$\alpha(\lambda) > 0 \quad \text{if } \lambda \in [0, 1), \quad \alpha(1) = 0. \tag{1.4}$$

In other words, the friction is deemed to be negligible when the fluid is in the vapor phase. In Section 2 we introduce the dynamical system we shall be dealing with and give a description of the main results. Detailed statements and proofs are provided in Sections 4–6, because system (1.1) shows up a so large variety of different traveling waves that it would be cumbersome to summarize all of them in a single statement. Section 3 contains some preliminary results and Section 7 investigates the uniqueness of constant solutions. We collect and comment all the results we obtained in a pictorial point of view in the final Section 8.

2. The dynamical system and an overview of the main results

We denote $U = (v, u, \lambda)$ and $\Omega = (0, +\infty) \times \mathbb{R} \times [0, 1]$. A traveling wave to (1.1) with constant speed c is a solution to (1.1) of the form

$$U(\xi) = U\left(\frac{x - ct}{\tau}\right),$$

satisfying

$$\begin{cases} -cv' - u' = 0, \\ -cu' + p' = -A(\lambda)u, \\ -c\lambda' = (p - p_e)\lambda(\lambda - 1), \end{cases} \tag{2.1}$$

together with

$$\begin{cases} (v, u, \lambda)(\pm\infty) = (v_\pm, u_\pm, \lambda_\pm), \\ (v', u', \lambda')(\pm\infty) = 0, \end{cases} \tag{2.2}$$

for $(v_\pm, u_\pm, \lambda_\pm) \in \Omega$. Here above, "′" denotes differentiation with respect to ξ, $p' = p_v v' + p_\lambda \lambda'$ and

$$A(\lambda) = \alpha(\lambda)\tau.$$

The end states $(v_\pm, u_\pm, \lambda_\pm)$ in the first line of (2.2) must be equilibrium points of (2.1) because of the second line in (2.2); the equilibrium set of (2.1) is

$$E = \{(v, 0, \lambda) \in \Omega : (p - p_e)\lambda = 0\} \cup \{(v, u, 1) \in \Omega\} = E_0 \cup E_1. \tag{2.3}$$

The case when the end states belong to $E_0 \cup \{E_1 \cap \{u = 0\}\}$ is contained in [14,15]. Then, we focus on the case that

exactly one of the end states belongs to $E_1 \cap \{u \neq 0\}$. (2.4)

If a solution to (2.1)–(2.2) exists, we say that $U_- = (v_-, u_-, \lambda_-) \to (v_+, u_+, \lambda_+) = U_+$ is a connection with speed c. We focus our analysis on the case

$$c > 0. \tag{2.5}$$