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Multiple positive solutions for a critical elliptic system with concave and convex nonlinearities☆

Haining Fan *

College of Science, China University of Mining and Technology, Xuzhou 221116, China

HIGHLIGHTS

- We get a relation about the positive solutions of our problem.
- Our problem is involving critical nonlinearities.
- Our result is available for sub-linear term $|u|^{q-2}u$ with all $1 < q < 2$.

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ABSTRACT

In this paper, we study the multiplicity results of positive solutions for a semi-linear elliptic system involving both concave–convex and critical growth terms. With the help of Nehari manifold and Lusternik–Schnirelmann category, we prove that the problem admits at least $cat(\Omega) + 1$ distinct positive solutions.

1. Introduction and the main result

In this paper, we are concerned with the number of positive solutions of the elliptic system:

$$(E_{\lambda,\mu})\begin{cases} -\Delta u = \lambda|u|^{q-2}u + \dfrac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^{\beta}, & \text{in } \Omega, \\ -\Delta v = \mu|v|^{q-2}v + \dfrac{2\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v, & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary, $N \geq 3, 1 < q < 2, \lambda, \mu > 0, \alpha, \beta > 1$ satisfy $\alpha+\beta = 2^* = \frac{2N}{N-2}$, which is the critical exponent.

Elliptic systems have extensive practical backgrounds and have received much attention recently. For example, they can be used to describe the multiplicative chemical reaction catalyzed by the catalyst grains under constant or variable temperature, a correspondence of the stable station of dynamical system determined by the reaction–diffusion system, see Ladde, Lakshmikantham and Vatsala in [1]. More naturally, it can be the populations by competing species in [2]. Therefore, much attention has been paid to the existence of nontrivial solutions for non-variational systems, potential systems and hamiltonian systems, see for instance, [3–7] and their references.

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* Tel.: +86 02762871301; fax: +86 02762871301.
E-mail address: fanhaining888@163.com.

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In particular, for the systems of semi-linear elliptic equations with concave–convex nonlinearities, various studies concerning the solutions structures have been presented [8–12]. Among these, Hsu [8] proved that $(E_{\lambda,\mu})$ permits at least two positive solutions when the pair of parameters (λ, μ) belongs to a certain subset of $\mathbb{R}^2$. Similar results were obtained by Adriouch and EL Hamidi [9]. Further studies involving sign-changing weight functions were taken by Wu [10] and Chen and Wu [11] for example, where the multiplicity results were obtained for the subcritical case $2 < \alpha + \beta < 2^*$ in [10] while these for the critical case $\alpha + \beta = 2^*$ were obtained in [11]. Their tool is the decomposition of the Nehari manifold.

In this work we aim to get better information on the number of positive solutions of $(E_{\lambda,\mu})$, for $\lambda, \mu > 0$ small enough, via the tools of the variational theory and the Lusternik–Schnirelmann category theory. When $q \geq 2$, employing the Lusternik–Schnirelmann category, it was shown in [13] that if $0 \in \Omega$, $N \geq 4$ and $2 \leq q \leq 2^*$, for $\lambda, \mu > 0$ small enough $(E_{\lambda,\mu})$ has at least $cat(\Omega)$ positive solutions, where $cat(\Omega)$ denotes the Lusternik–Schnirelmann category of Ω in itself. For similar results, we refer the reader to [14–16]. When $1 < q < 2$, S. Benmouloud, R. Echarghaoui, and S.M. Sbai [12] recently showed that $(E_{\lambda,\mu})$ admits at least $cat(\Omega) + 1$ positive solutions under the condition that $0 \in \Omega$, $N > 4$, $2^* - \frac{N}{N-2} \leq q < 2$ and $\lambda, \mu > 0$ are sufficiently small. In this paper, by borrowing some techniques of [11] and the arguments developed in [17], we give a similar result as in [12] but under weaker conditions in the following.

Theorem 1.1. *Assume $1 < q < 2$. Then there exists $\Lambda_* > 0$ such that if $\lambda, \mu \in (0, \Lambda_*)$, $(E_{\lambda,\mu})$ has at least $cat(\Omega) + 1$ distinct positive solutions.*

Remark 1.1. If Ω is a general domain, $cat(\Omega) \geq 1$ and Theorem 1.1 is the main result of [8].

This paper is organized as follows: In Section 2, we give some notations and preliminary results. In Section 3, we discuss some concentration behavior. In Section 4, we prove Theorem 1.1.

2. Notations and preliminaries

We propose to study $(E_{\lambda,\mu})$ in the framework of the Sobolev space $H = H_0^1(\Omega) \times H_0^1(\Omega)$ using the standard norm

$$\|(u, v)\|_H = \left(\int_\Omega (|\nabla u|^2 + |\nabla v|^2)dx\right)^{\frac{1}{2}}.$$

Denote

$$S_{\alpha,\beta} := \inf_{u,v \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega (|\nabla u|^2 + |\nabla v|^2)dx}{\left(\int_\Omega |u|^\alpha |v|^\beta dx\right)^{\frac{2}{\alpha+\beta}}}.$$

Working as in the proof of [18, Theorem 5], we deduce that

$$S_{\alpha,\beta} = \left(\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} + \left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}\right) S,$$

where S is the best Sobolev constant, that is

$$S := \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\left(\int_\Omega |u|^{2^*} dx\right)^{\frac{2}{2^*}}}.$$

It is well known that S is independent of Ω, and for each $\varepsilon > 0$,

$$v_\varepsilon(x) = \frac{[N(N-2)\varepsilon^2]^{(N-2)/4}}{(\varepsilon^2 + |x|^2)^{(N-2)/2}}$$

is a positive solution of the critical problem

$$-\Delta u = |u|^{2^*-2}u \quad \text{in } \mathbb{R}^N$$

with $\int_{\mathbb{R}^N} |\nabla v_\varepsilon|^2 dx = \int_{\mathbb{R}^N} |v_\varepsilon|^{2^*} dx = \frac{1}{N} S^{N/2}$.

Positive solutions to $(E_{\lambda,\mu})$ will be obtained as critical points of the corresponding energy functional

$$I_{\lambda,\mu}(u, v) = \frac{1}{2}\|(u, v)\|_H^2 - \frac{1}{q}\int_\Omega (\lambda(u_+)^q + \mu(v_+)^q)dx - \frac{2}{\alpha+\beta}\int_\Omega (u_+)^\alpha (v_+)^\beta dx,$$

where $u_+ = \max\{u, 0\}$ and $v_+ = \max\{v, 0\}$. From the assumption, it is easy to prove that $I_{\lambda,\mu}$ is well defined in H and $I_{\lambda,\mu} \in C^2(H, \mathbb{R})$. Then we consider the behaviors of $I_{\lambda,\mu}$ on the Nehari manifold

$$N_{\lambda,\mu} = \{u, v \in H_0^1(\Omega)\setminus\{0\}; I'_{\lambda,\mu}(u, v)(u, v) = 0\}.$$