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On a cross-diffusion segregation problem arising from a model of interacting particles ☆

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ABSTRACT

We prove the existence of solutions of a cross-diffusion parabolic population problem. The system of partial differential equations is deduced as the limit equations satisfied by the densities corresponding to an interacting particles system modeled by stochastic differential equations. According to the values of the diffusion parameters related to the intra and inter-population repulsion intensities, the system may be classified in terms of an associated matrix. For proving the existence of solutions when the matrix is positive definite, we use a fully discrete finite element approximation in a general functional setting. If the matrix is only positive semi-definite, we use a regularization technique based on a related cross-diffusion model under more restrictive functional assumptions. We provide some numerical experiments demonstrating the weak and strong segregation effects corresponding to both types of matrices.

1. Introduction

The effects of spatial cross-diffusion on interacting population models have been widely studied since Kerner [1] and Jorné [2] examined the linear cross-diffusion model

$$\partial_t u_i - a_{i1}\Delta u_1 - a_{i2}\Delta u_2 = (-1)^{i+1} u_i(\alpha_i - \beta_i u_j),$$

with non-negative self-diffusivities a_{ii}, and non-zero cross-diffusivities a_{ij}, for $i, j = 1, 2,\ i \neq j$, and demonstrated that while self-diffusion tends to damp out all spatial variations in the Lotka–Volterra system, cross-diffusion may give rise to instabilities [3] and to non-constant stationary solutions.

First nonlinear cross-diffusion models seem to have been introduced by Busenberg and Travis [4] (see also Gurtin and Pipkin [5] for a related model), and Shigesada et al. [6] from different modeling points of view. Shigesada et al. approach starts with the assumption of a single population density evolution determined by a continuity equation

$$\partial_t u - \text{div} J(u) = u(\alpha - \beta u), \quad \text{with } J(u) = \nabla((c + au)u) + bu\nabla\Phi. \tag{1}$$

The divergence of the flow J is thus decomposed into three terms: a random dispersal, $c\Delta u$, a dispersal caused by *population pressure*, $a\Delta u^2$, and a drift directed to the minima of the environmental potential Φ. Generalizing this scalar equation to two populations they propose the system, for $i = 1, 2$,

$$\partial_t u_i - \text{div} J_i(u_1, u_2) = f_i(u_1, u_2),$$

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with

$$J_i(u_1, u_2) = \nabla\left((c_i + a_{i1}u_1 + a_{i2}u_2)u_i\right) + b_i u_i \nabla\Phi, \tag{2}$$

and f_i of the competitive Lotka–Volterra type. Disregarding the linear dispersals ($c = c_i = 0$) representing a random contribution to the motion, the nonlinear part of the flow J in Eq. (1) may be expressed in the conservative form as $J(u) = u\tilde{J}(u)$, with $\tilde{J}$ given by the potential $\tilde{J}(u) = \nabla(2au + b\Phi)$. However, rewriting the flows (2) in a similar way leads to the more intricate expression

$$\tilde{J}_i(u_1, u_2) = \left(2a_{ii} + a_{ij}\frac{u_j}{u_i}\right)\nabla u_i + a_{ij}\nabla u_j + b_i\nabla\Phi,$$

which, in general, cannot be deduced from a potential. This fact has been one of the main difficulties in finding appropriate conditions ensuring the existence of solutions to the model proposed by Shigesada et al. (*SKT model*, from now on), see [7–16] and their references.

The generalization of the flow in (1) to several populations (with $c = b = 0$) given by Busenberg and Travis [4] is perhaps more natural from the modeling point of view. They assume that the individual population flow J_i is proportional to the gradient of a potential function, Ψ, that only depends on the total population density $U = u_1 + u_2$,

$$J_i(u_1, u_2) = a\frac{u_i}{U}\nabla\Psi(U).$$

Note that in this way the flow of U is still given in the form (1), with $J(U) = a\nabla\Psi(U)$ (and $c = b = 0$). Assuming the power law $\Psi(s) = s^2/2$, we obtain individual population flows given by

$$J_i(u_1, u_2) = au_i\nabla U, \tag{3}$$

as those introduced by Gurtin and Pipkin [5] and mathematically analyzed by Bertsch et al. [17,18].

In this article we propose a generalization of the Busenberg–Gurtin model consisting of the assumption that the individual flows J_i depend, instead of on the total population density $u_1 + u_2$, on a general linear combination of both population densities, possibly different for each population. As remarked in [5], these weighted sums are motivated when considering a set of species with different characteristics, such as size, behavior with respect to overcrowding, etc. In addition, we also assume that the flows may contain environmental and random effects, which altogether lead to the following form

$$J_i(u_1, u_2) = u_i\nabla(a_{i1}u_1 + a_{i2}u_2 + b_i\Phi) + c_i\nabla u_i,$$

which (for $c_i = 0$) has a conservative form similar to that of the scalar case. We shall refer to this model as the *BT model*.

Let us finally remark that cross-diffusion parabolic systems have been used to model a variety of phenomena ranging from ecology [19–21,16,22–24], to semiconductor theory [25,26], granular materials [27–29] or turbulent transport in plasmas [30], among others. Apart from the global existence and regularity results for the evolution problem, construction of traveling wave solutions [31] or exact solutions [32] has been accomplished. For the steady state problem, the existence of non constant steady state solutions has been proven in [33,10]. Other interesting properties, such as pattern formation, has been studied in [34–38]. Finally, the numerical discretization has received much attention, and several schemes have been proposed [11,12,14,39–41].

The article is organized as follows. In Section 2, for a better physical understanding of our model, we sketch a heuristic deduction based on stochastic dynamics of particle systems. In Section 3 we give the precise assumptions on the data problem and state the main results. In Section 4, we introduce the approximated problems and perform some numerical experiments showing the behavior of solutions under several choices of the parameters, including a comparison between the SKT and the BT models. In Section 5, we prove the theorems stated in Section 3, finally, in Section 6 we present our conclusions.

2. Mathematical modeling

In recent years there has been a trend to the rigorous deduction of Eq. (1) as the equation satisfied by the limit density distribution of suitable particle stochastic systems of differential equations, see [42–45] and their references. We sketch here the formulation and the main ideas contained in these works which allow us to deduce our model.

Consider a system of $N = N_1 + N_2$ interacting particles of two different types described by their trajectories $X^i_{j_i} : \mathbb{R}_+ \to \mathbb{R}^m$, $j_i = 1, \dots, N_i$, $i = 1, 2$ (stochastic processes). We take $N_1 = N_2 = n$ to simplify the notation. The Lagrangian approach to the description of the system is based on specifying suitable interacting laws among particles in such a way that their trajectories are determined by solving the following stochastic system of ordinary differential equations (SDE)

$$dX^i_j(t) = F^i_j(X^1_1(t), \dots, X^1_n(t), X^2_1(t), \dots, X^2_n(t))dt + \sigma^i_n dW^i_j(t), \tag{4}$$

together with some initialization of the processes $X^i_j(0) = X^i_{j0}$, $j = 1, \dots, n$, $i = 1, 2$. Functions $F^i_j : \mathbb{R}^{2n} \to \mathbb{R}^m$ describe deterministic interactions among particles while the constants σ^i_N are the intensities of random dispersal, due to a variety of