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Some nonlinear internal equatorial flows

Hung-Chu Hsu *

Department of Mathematics, King's College London, Westminster, UK
Tainan Hydraulics Laboratory, National Cheng Kung University, Tainan 701, Taiwan

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ABSTRACT

We present an exact solution of the nonlinear governing equations for geophysical water waves in the β-plane approximation near the equator.

1. Introduction

Recently, some exact solutions describing nonlinear equatorial flows in the Lagrangian framework were obtained. In Constantin [1] equatorially trapped wind waves were presented—see also the discussion in Constantin & Germain [2], and Henry [3] showed that one can also include a uniform underlying current. In Constantin [4] internal waves describing the oscillation of the thermocline as a density interface separating two layers of constant density, with the lower layer motionless, were presented. Our aim is to extend the solution in Constantin [4] to include an underlying uniform current. The presence of strong currents in the Equatorial Pacific is well-documented, cf. Philander [5]. The present extension of the flow in Constantin [4], while showing that one can accommodate an underlying current, differs from the flow presented recently in Constantin [6].

2. Preliminaries

The Earth is taken to be a sphere of radius, $R = 6371$ km, rotating with constant rotational speed $\Omega = 7.29 \cdot 10^{-5}$ rad s^{-1} round the polar axis toward the east, in a rotating frame with the origin at a point on the earth's surface, so that the Cartesian coordinates $(x,\ y,\ z)$ represent longitude, latitude, and the local vertical, respectively. The governing equations for geophysical ocean waves are, cf. Pedlosky [7], the Euler equation

$$\begin{cases} u_t + uu_x + vu_y + wu_z + 2\Omega w\cos\phi - 2\Omega v\sin\phi = -\dfrac{1}{\rho}P_x, \\ v_t + uv_x + vv_y + wv_z + 2\Omega u\sin\phi = -\dfrac{1}{\rho}P_y, \\ w_t + uw_x + vw_y + ww_z - 2\Omega u\cos\phi = -\dfrac{1}{\rho}P_z - g, \end{cases} \tag{1}$$

coupled with the equation of mass conservation

$$\rho_t + u\rho_x + v\rho_y + w\rho_z = 0 \tag{2}$$

and with the incompressibility constraint

$$u_x + v_y + w_z = 0. \tag{3}$$

* Correspondence to: Tainan Hydraulics Laboratory, Tainan, Taiwan. Tel.: +886 6 2371938.
E-mail address: hchsu@thl.ncku.edu.tw.

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Here t stands for time, ϕ for latitude, $g = 9.81\,\mathrm{m\,s^{-2}}$ is the (constant) gravitational acceleration at the Earth's surface, and ρ is the water's density, while P is the pressure.

Since we restrict our attention to a symmetric band of width of about 250 km on each side of the Equator, the approximations $\sin\phi \approx \phi$ and $\cos\phi \approx 1$ can be used, cf. Vallis [8]. This approximation, called the equatorial β-plane approximation, approximates the Coriolis force

$$2\Omega \begin{pmatrix} w\cos\phi - v\sin\phi \\ u\sin\phi \\ -u\cos\phi \end{pmatrix}$$

by

$$\begin{pmatrix} 2\Omega w - \beta y v \\ \beta y u \\ -2\Omega u \end{pmatrix}$$

with $\beta = 2\Omega/R = 2.28 \cdot 10^{-11}\ \mathrm{m^{-1}\,s^{-1}}$, cf. Cushman-Roisin and Becker [9]. Consequently, the Euler equation (1) is replaced by

$$\begin{cases} u_t + uu_x + vu_y + wu_z + 2\Omega w - \beta y v = -\dfrac{1}{\rho}P_x, \\ v_t + uv_x + vv_y + wv_z + \beta y u = -\dfrac{1}{\rho}P_y, \\ w_t + uw_x + vw_y + ww_z - 2\Omega u = -\dfrac{1}{\rho}P_z - g. \end{cases} \tag{4}$$

We work with a two-layer model: two layers of constant densities, separated by a sharp interface—the thermocline. Let $z = \eta(x, y, t)$ be the equation of the thermocline. We model the oscillations of this interface as propagating in the longitudinal direction at constant speed c. The upper boundary of the region $M(t)$ above the thermocline and beneath the near-surface layer $L(t)$ to which wind effects are confined is given by $z = \eta_+(x, y, t)$. Beneath the thermocline the water has a constant density ρ_+ and is still: at every instant t we have $u = v = w = 0$ for $z < \eta(x, y, t)$. From (4) we infer that

$$P = P_0 - \rho_+ g z \quad \text{in the region } z < \eta(x, y, t),$$

for some constant P_0. We investigate the eastward propagation of geophysical waves with vanishing meridional velocity ($v \equiv 0$) in the region $M(t)$, without discussing the interaction of geophysical waves and wind waves in the region $L(t)$. Throughout $M(t)$ the water is assumed to have constant density $\rho_0 < \rho_+$, the typical value of the reduced gravity

$$\tilde{g} = g\,\frac{\rho_+ - \rho_0}{\rho_0} \tag{5}$$

being $6 \cdot 10^{-3}\ \mathrm{m\,s^{-2}}$, cf. Fedorov and Brown [10]. Consequently, we seek solutions ($u(x, y, z, t)$, $w(x, y, z, t)$, $\eta(x, y, t)$ and $\eta_+(x, y, t)$) of the Euler equations in the form

$$\begin{cases} u_t + uu_x + wu_z + 2\Omega w = -\dfrac{1}{\rho}P_x, \\ \beta y u = -\dfrac{1}{\rho}P_y, \\ w_t + uw_x + ww_z - 2\Omega u = -\dfrac{1}{\rho}P_z - g, \end{cases} \tag{6}$$

in $\eta(x, y, t) < z < \eta_+(x, y, t)$, coupled with the incompressibility condition

$$u_x + w_z = 0 \quad \text{in } \eta(x, y, t) < z < \eta_+(x, y, t), \tag{7}$$

and with the boundary condition

$$P = P_0 - \rho_+ g z \quad \text{on } z = \eta(x, y, t). \tag{8}$$

Moreover, we impose that the flow approaches a uniform current rapidly in the near-surface layer, that is

$$(u, v) \to (-U, 0) \quad \text{as } z \to \eta_+(x, y, t). \tag{9}$$