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Double obstacle control problem for a quasilinear elliptic variational inequality with source term

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ABSTRACT

In this paper, taking the double obstacle as the control variable, we consider an optimal control problem for a quasilinear elliptic variational inequality with source term. We establish the existence theorem and derive the optimality system for the underlying problem.

1. Introduction

Suppose $\Omega \subset \mathcal{R}^n$ is a bounded domain with a $C^{1,1}$ boundary $\partial\Omega$. Let $z_d \in L^2(\Omega)$ be a given target profile. For any $\varphi, \psi \in W_0^{1,p}(\Omega)$, we define

$$K(\varphi, \psi) = \{z \in W_0^{1,p}(\Omega) | \varphi \le z \le \psi \text{ a.e. } x \in \Omega\}$$

and consider the following quasilinear elliptic double obstacle variational inequality with source term:

$$\begin{cases} y \in K(\varphi, \psi), \\ \displaystyle\int_\Omega A(x, \nabla y) \cdot \nabla(z - y)dx \ge \int_\Omega f(x, y)(z - y)dx, \quad \forall z \in K(\varphi, \psi) \end{cases} \tag{1.1}$$

where

$$A(x, \eta) = (a_1(x, \eta), \dots, a_n(x, \eta)). \tag{1.2}$$

In practice, there are many real physical, geometrical and financial problems related to obstacle variational inequalities (cf. [1–3]). Except some ideal cases, the governing equations are usually quasilinear or nonlinear. A typical example is that in the study of non-Newtonian fluids [4]

$$A(\nabla y) = |\nabla y|^{p-2}\nabla y.$$

Another example is that in the study of minimal surface with obstacle [3,5]

$$A(\nabla y) = (1 + |\nabla y|^2)^{-\frac{1}{2}}\nabla y.$$

Similar cases occur in the evolutionary problems. From both theoretical and practical points of view, it is necessary to study the case with $A(x, \cdot)$ nonlinear.

Given $\varphi, \psi \in W_0^{1,p}(\Omega)$, under some further mild assumptions on $A(x, \eta)$ and $f(x, y)$ (see (H_1)–(H_3) below), the variational inequality (1.1) is uniquely solvable [6]. We will denote the unique solution of (1.1) corresponding to (φ, ψ) by $y = T(\varphi, \psi)$.

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The variational inequalities and related optimal control problems have been studied extensively for decades (see [1–3,5,7–31] and the reference therein). Recently, various control problems for double obstacle variational inequalities have been considered in [11,16–18]. When the governing system is an obstacle variational inequality, the obstacle itself can also be regarded as the control. Such a case is referred to as an obstacle optimal control problem. To the best of our knowledge, such problem was first studied in [7]. For the homogeneous case, an optimal obstacle control problem for an elliptic variational inequality is considered there. To study the obstacle optimal control problem for more general system, an indirect obstacle control model is suggested in [14,15]. Motivated by [7,14], the regularity of the obstacle control problem has been investigated in [25,26]. The work in [7] has been extended in [8] by adding a non-zero source term to the right hand side of the state equation, and it has been found that such an extension is not trivial. However, the techniques developed in [7,8] do not work when $A(x,\cdot)$ is nonlinear.

Let

$$W = W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$$

and

$$U_{ad} = \{(\varphi,\psi)\in W\times W|\varphi\le\psi \text{ a.e. } \Omega\}.$$

We seek a pair of $(\bar\varphi,\bar\psi)\in U_{ad}$ so that the corresponding state $\bar y = T(\bar\varphi,\bar\psi)$ is close to the desired target profile z_d and the norm of $(\bar\varphi,\bar\psi)$ is not too large in $W\times W$, i.e., we try to minimize the objective functional

$$J(\varphi,\psi) = \int_\Omega \left\{\frac{1}{2}(T(\varphi,\psi)-z_d)^2 + \frac{1}{p}[|\Delta\varphi|^p + |\Delta\psi|^p]\right\} dx. \tag{1.3}$$

More precisely, in this paper we study the following optimal control problem:

Problem (P). Find a pair of control $(\bar\varphi,\bar\psi)\in U_{ad}$, such that

$$J(\bar\varphi,\bar\psi) = \inf_{U_{ad}} J(\varphi,\psi). \tag{1.4}$$

Throughout the paper, we assume $p>\max\{n,2\}$ is a given number, and make the following assumptions on $A(x,\eta)$ and $f(x,y)$:

(H_1) For any $\eta=(\eta_1,\ldots,\eta_n)\in\mathcal{R}^n$, $a_j(\cdot,\eta)$ is a measurable function on Ω with $a_j(\cdot,0)=0$ and for any $x\in\Omega$, $a_j(x,\cdot)$ belongs to $C^1(\mathcal{R}^n)$, $j=1,\ldots,n$.

(H_2) For any $x\in\Omega$ and all $\xi,\eta\in\mathcal{R}^n$

$$\sum_{i,j=1}^n \frac{\partial a_j}{\partial\eta_i}(x,\eta)\xi_i\xi_j \ge \Lambda_1(\lambda+|\eta|)^{p-2}|\xi|^2,$$

$$\sum_{i,j=1}^n \left|\frac{\partial a_j}{\partial\eta_i}(x,\eta)\right| \le \Lambda_2(\lambda+|\eta|)^{p-2},$$

where $\lambda\in(0,1]$, Λ_1 and Λ_2 are some positive constants.

(H_3) For any $y\in\mathcal{R}$, $f(\cdot,y)$ is a measurable function on Ω with $f(\cdot,0)=0$. For any $x\in\Omega$, the function $y\mapsto f(x,y)$ is decreasing, belongs to $C^1(\mathcal{R})$ and

$$|f(x,y)|\le\Lambda_3|y|^{p-1}\quad\forall(x,y),$$

where Λ_3 is a positive constant.

Some specific functions $A(x,\eta)$ and $f(x,y)$ which satisfy (H_1)–(H_3) can be found in [32].

The following lemma is an immediate consequence of assumptions (H_1) and (H_2).

Lemma 1 (*Cf.* [32]). *Under Assumptions* (H_1)–(H_2), *there are positive constants* k_1 *and* k_2 *depending only on* n, Λ_1 *and* Λ_2 *such that for any* $x\in\Omega$, $\eta=(\eta_1,\ldots,\eta_n)\in\mathcal{R}^n$ *and* $\eta'=(\eta_1',\ldots,\eta_n')\in\mathcal{R}^n$

(a)

$$\sum_{j=1}^n (a_j(x,\eta)-a_j(x,\eta'))(\eta_j-\eta_j')\ge k_1|\eta-\eta'|^p.$$

(b)

$$\sum_{j=1}^n |a_j(x,\eta)|\le k_2|\eta|^{p-1}.$$

In this paper, we need some basic results in real analysis. Here, for reader's convenience, we list the following lemmas without proof.

Lemma 2 (*Cf.* [33]). *Let* $1<p<\infty$, $\{g_n\}$ *is bounded in* L^p, *and* $g_n\to g$, *a.e. in* Ω. *Then*

$$g_n \xrightarrow{w} g \quad \text{in } L^p(\Omega).$$