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Remarks on traveling waves and equilibria in fluid dynamics with viscosity, capillarity, and heat conduction

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ABSTRACT

Heat conduction causes a tough obstacle in studying traveling waves in fluid dynamics. In this note we consider the fluid dynamics equations where viscosity, capillarity and heat conduction coefficients are present. First we transform the model into the one with an equation for the entropy as the conservation of energy. Then, given any traveling wave of the viscous–capillary–heat conductive model connecting two given states, we derive a corresponding system of differential equations. Then we show that this system of differential equations possesses the equilibria which correspond to the two states of the given traveling wave. This work may therefore motivate future study to solve challenging open questions on the stability of these equilibria and the existence of the traveling waves in fluid dynamics with heat conduction.

1. Introduction

We are interested in the traveling waves of the following model of fluid dynamics equations with viscosity, capillarity, and heat conduction

$$
\begin{aligned}
&v_t - u_x = 0,\\
&u_t + p_x = \left(\frac{\lambda}{v}u_x\right)_x - (\mu v_x)_{xx} + \left(\frac{\mu_v}{2}v_x^2\right)_x,\\
&E_t + (up)_x = \left(\frac{\lambda}{v}uu_x\right)_x + \left(\frac{\mu_v}{2}uv_x^2 - u(\mu v_x)_x\right)_x + (\mu u_x v_x)_x + \left(\frac{\kappa}{v}T_x\right)_x,
\end{aligned}
\tag{1.1}
$$

for $x \in \mathbb{R}$ and $t > 0$. Here, v, S, p, ε, T denote the specific volume, entropy, pressure, internal energy, temperature, respectively; u is the velocity, and

$$
E = \varepsilon + \frac{u^2}{2} + \frac{\mu}{2}v_x^2 \tag{1.2}
$$

is the total energy. The non-negative quantities λ, μ, κ represent the viscosity, capillarity, and the heat conduction, respectively. In general, these quantities can be considered as functions of the thermodynamic variables. One may therefore write $\lambda = \lambda(v, S)$, $\mu = \mu(v, S)$ and $\kappa = \kappa(v, S)$ when choosing v, S as the independent thermodynamic variables. Here, the Lagrangian coordinates are chosen so that the calculations are simple only, since similar results hold for the Eulerian coordinates.

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A shock wave connecting a left-hand state U_- and a right-hand states U_+ can be seen as admissible if it is obtained as the point-wise limit by vanishing additional terms involving higher-order derivatives such as viscosity, capillarity, and heat conduction, see [1]. Therefore, traveling waves can be used to justify an admissibility criterion for shock waves of different types, such as the Lax shocks [2] and nonclassical shocks [3]. However, the study of traveling waves, though it has attracted many authors, has been restricted mostly to the case where additional terms are merely viscosity and capillarity. Heat conduction is commonly ignored. The obstacle for the study of traveling waves when heat conduction is present is that this quantity causes serious inconvenience to establish a system of ordinary differential equations whose equilibria are used to characterize the traveling waves. In this work, we propose a "complete" model that contains not only the viscosity and capillarity, but also the heat conduction. It turns out that the capillarity coefficient is useful to deal with the heat conductive term. First, we transform system (1.1) into a new one where the conservation of energy is expressed by an equation of the entropy. Second, this new system is suitable to establish a system of ordinary differential equations whose equilibria correspond to the left-hand and right-hand states of a given traveling wave. Note that these two states are also the states of a shock wave approximated by the traveling wave. It is interesting to note also that unlike most results known in the literature, the system of ordinary differential equations depends on the type of fluid. This explains the presence of the heat conduction, where the temperature is characterized by equations of state of the fluid under consideration and reflects the type of the fluid. We choose the three most popular types of fluid to establish the corresponding system of ordinary differential equations: the ideal fluids, the stiffened gases, and the van der Waals fluids. Nevertheless, the corresponding system of differential equations for fluids of other types can be derived in a similar manner. This work thus raises open questions for future study on the stability of the equilibria and the existence of the traveling waves.

There have been many studies on traveling waves in the literature. A pioneering work on the shock layers of the gas dynamics equations with viscosity and heat conduction effects (with zero capillarity) was presented in [4]. Traveling waves for diffusive–dispersive scalar equations were earlier considered in [5,6]. Traveling waves of the hyperbolic–elliptic model of phase transition dynamics were also studied in [7–11]. Traveling waves in Korteweg models in the isothermal case, the Eulerian and Lagrangian capillarity models were also studied in [12–14]. The existence of traveling waves associated with Lax shocks for viscous–capillary models was considered in our recent works [15–19]. These works developed the method of estimating the attraction domain of the asymptotically stable equilibrium point to establish the existence of traveling waves. The existence of traveling waves corresponding to nonclassical shocks for viscous–capillary models was considered in [3,20–25]. See also the references therein.

The organization of this paper is as follows. Section 2 provides basic concepts and properties of the fluid dynamics equations. In Section 3 we present the derivation of an equivalent system to (1.1) where the conservation of energy is given as an equation of the entropy. In Section 4 we establish the autonomous system of ordinary differential equations for each given traveling wave. It is shown that the equilibria of this system correspond to the left-hand and right-hand states of the given traveling wave.

2. Preliminaries

Consider the fluid dynamics equations in the Lagrangian coordinates

$$\begin{aligned}&v_t - u_x = 0,\\&u_t + p_x = 0,\\&E_t + (up)_x = 0, \quad x \in \mathbb{R}, t > 0.\end{aligned} \tag{2.1}$$

A shock wave of (2.1) is a weak solution U of the form

$$U(x,t) = \begin{cases} U_-, & \text{if } x < st,\\ U_+, & \text{if } x > st,\end{cases} \tag{2.2}$$

where U_- and U_+ are constant, known as the left-hand and right-hand states, and a constant s, known as the shock speed. These values must satisfy the Rankine–Hugoniot relations

$$\begin{aligned}&s(v_+ - v_-) + (u_+ - u_-) = 0,\\&-s(u_+ - u_-) + p_+ - p_- = 0,\\&\varepsilon_+ - \varepsilon_- + \frac{p_+ + p_-}{2}(v_+ - v_-) = 0.\end{aligned} \tag{2.3}$$

The last equation of (2.3) is also known as the Hugoniot equation. The shock speed s can be given from (2.3) as

$$s = s(U_-, U_+) = \pm\sqrt{-\frac{p_+ - p_-}{v_+ - v_-}}, \tag{2.4}$$

provided

$$\frac{p_+ - p_-}{v_+ - v_-} \le 0.$$