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Some remarks on topological horseshoes and applications

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ABSTRACT

An investigation on topological methods proving chaotic dynamics is presented: the relationships between Kennedy, Koçak and Yorke's "chaos lemma" and Medio, Pireddu and Zanolin's "stretching along the paths" put in evidence a double way to prove that a discrete dynamical system is chaotic according to Block and Coppel's definition of chaos.

Particular relevance is given to non-injective discrete systems, such as Lotka–Volterra with Holling Type II, since they are strongly involved in semi-conjugacy.

1. Introduction and motivations

The mathematical study of *dynamical systems and chaos* has involved not only mathematicians but also physicists, science engineers and philosophers of science since the second half of 1800. Though it is not easy to assign the origin of *chaos theory* as a mathematical subject, it is quite unanimously recognized that an early proponent was Henri Poincaré with the *three body problem* (see [1]).

Despite of its far origins, mathematical definitions of chaos have appeared only in the last 40 years. The first one – in chronological order – was given by Li and Yorke in their celebrated article *Period three implies chaos*. According to the authors, a map f defined on a compact metric space is chaotic if it has k-periodic points for every natural number k and satisfies three more conditions concerning the asymptotic behavior of the trajectories (see [2]). Li and Yorke's article has become fundamental because they showed that a map having a 3-period cycle has also all-period cycles.

Another important definition was given by Devaney in 1986 in [3]. He described the chaotic behavior under three geometrical and analytical conditions: sensitivity to initial conditions, density of periodic points and topological transitivity. In some cases (e.g. taking the functions defined on a real interval) the first two properties are superfluous, so that topological transitivity has become the most relevant property. Indeed it has been investigated by many other authors: some of them found out similar conditions (Crannell in [4]), others instead put together sensitivity and transitivity in only one condition (Touhey in [5]).

In 1992 Block and Coppel established a topological approach to define a chaotic map (see [6]). We will deeply analyze their definition in the sequel as we come to the concepts of symbolic dynamics and semi-conjugacy (paragraphs 2.3 and 2.4) that are very common in chaotic dynamics.

Roughly speaking, they considered a set – called Σ_2^+ – as the space of infinite sequences with entries 0 or 1. Formally, a typical element of Σ_2^+ is $\{s_0 s_1 s_2 \cdots s_k \cdots\}$, where $s_i = 0$ or $s_i = 1$ for every natural number i.

They defined a metric $d(\cdot, \cdot)$ on this space and a particular function: the well known *shift map* (or *Bernoulli shift map*) $\sigma : \Sigma_2^+ \to \Sigma_2^+$ such that

$$\sigma(\{s_0 s_1 s_2 \cdots s_k \cdots\}) = \{s_1 s_2 \cdots s_k \cdots\}.$$

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It is easy to prove that the shift map is continuous and surjective (see [3]). This map has turned out to be paradigmatic in chaos theory (since it is characterized by transitivity, positive entropy, sensitivity, etc.). Moreover, this construction offers a strategy to understand whether a system f is chaotic or not since it establishes a relation between f itself and the shift map σ (typically this relationship is semi-conjugacy or conjugacy) (see also [7,8]).

Formally Block and Coppel gave the following definition of chaos

Definition 1.1. Let X be a compact metric space, $f : X \to X$ a continuous function. Then f is *chaotic in the sense of Block and Coppel* (or *B/C-chaotic*) if there are an integer $m \in \mathbb{N}$ and a subset $Y \subseteq X$ such that $f^m|_Y$ is semi-conjugated to the Bernoulli shift σ defined on Σ_2^+.

In other words, a map is *B/C-chaotic* if there is a continuous and surjective map $\gamma : Y \to \Sigma_2^+$ such that the following diagram

$$\begin{array}{ccc} Y & \xrightarrow{f} & Y \\ \gamma\downarrow & & \downarrow\gamma \\ \Sigma_2^+ & \xrightarrow[\sigma]{} & \Sigma_2^+ \end{array}$$

is commutative. Block and Coppel's definition of chaos is strongly related with another one: if $m = 1$ in the Definition 1.1 then this notion of chaos is also known as *chaos in the sense of coin tossing* (see [9]).

Moreover, their original definition involved the topological concept of *turbolence* (see [10]) for planar maps; then it has been generalized (as in Definition 1.1) for a wider range of maps.

Though Li and Yorke's, Devaney's and Block and Coppel's definition come from different backgrounds, they have common properties: they are invariant under conjugacy and – under certain hypotheses – they are equivalent (see [11,9]).

In the literature we can find a much larger number of definitions: some of these have been created ad hoc for particular systems.

Since there is not a generally accepted definition, we would like to follow Block and Coppel's opinion: "*It is our view that any definition for more general spaces should agree with ours in the case of an interval. This requirement is not satisfied by some of the definitions used in literature*"(see [6] or [12] for bidimensional maps).

Furthermore, it is often quite difficult to prove directly that a dynamical system is chaotic according to Li and Yorke's or Devaney's definitions. Block and Coppel's offers a strategy to overcome this problem by establishing a relationship (semi-conjugacy) between the dynamical system itself and the shift map σ, which is known to be chaotic. In the literature indeed you can find many *B/C-chaotic* dynamical systems – even with stronger properties than semi-conjugacy to the shift map σ – as examples of this typical relationship: according to the example given by Devaney in [3], the logistic map, which is defined as follows

$$\begin{aligned} & f : \mathbb{R} \to \mathbb{R} \\ & f(x) = ax(1-x), \quad a \in \mathbb{R} \text{ and } a > 0 \end{aligned} \tag{1}$$

is conjugated (and not only semi-conjugated) to the Bernoulli shift, for $a > 2 + \sqrt{5}$. This means that – using the same notation of Devaney – there exist an invariant set $\Lambda \subset [0, 1]$ (which is a Cantor set) and a homeomorphism $S : \Lambda \to \Sigma_2^+$ such that the following diagram

$$\begin{array}{ccc} \Lambda & \xrightarrow{f} & \Lambda \\ S\downarrow & & \downarrow S \\ \Sigma_2^+ & \xrightarrow[\sigma]{} & \Sigma_2^+ \end{array} \tag{2}$$

is commutative.

Devaney proved that the logistic map is conjugated to the Bernoulli shift on Σ_2^+: this fact implies that it is also semi-conjugated to it, so that it is *B/C-chaotic* (see Remark 1.2 for a more formal explanation).

The logistic map is chaotic in the sense of Devaney (see [3]). Moreover it is also chaotic in Block and Coppel's one, which is related to a more general definition of chaos: this is the reason why in this article we will consider mainly Block and Coppel definition. To put it plainly, we can state that a system is chaotic because the properties of σ are preserved under semi-conjugacy (or conjugacy as well) as explained in [11]. Moreover, in this paper we will see that a system is *B/C-chaotic* if it satisfies some geometrical conditions (Theorems 2.2 and 2.3): in this way we will prove that a system satisfies *B/C-chaos* through an indirect way.

There are many chaotic systems in the sense of Block and Coppel, but one particular map has turned out to be a fundamental example of chaotic map: the *Smale's horseshoe*. This name derives from a particular planar map that Smale showed to be chaotic by a geometrical proof in [13,14]. More precisely, the author proved that this map and the Bernoulli full-shift on two symbols are topologically conjugated. We will develop the main features of the *horseshoe* following the approach