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Dynamical complexity induced by Allee effect in a predator–prey model

Weiming Wang [a,*], Ya-nuo Zhu [a], Yongli Cai [b], Wenjuan Wang [c]

[a] College of Mathematics and Information Science, Wenzhou University, Wenzhou, 325035, PR China

[b] School of Mathematics and Computational Science, Sun Yat-sen University, Guangzhou, 510275, PR China

[c] Department of Biostatistics and Computational Biology, University of Rochester Medical Center, 601 Elmwood Ave, Box 630, Rochester, NY 14642, USA

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ABSTRACT

In this paper, we investigate the complex dynamics induced by Allee effect in a predator–prey model. For the non-spatial model, Allee effect remains the boundedness of positive solutions, and it also induces the model to exhibit one or two positive equilibria. Especially, in the case with strong Allee effect, the model is bistable. For the spatial model, without Allee effect, there is the nonexistence of diffusion-driven instability. And in the case with Allee effect, the positive equilibrium can be unstable under certain conditions. This instability is induced by Allee effect and diffusion together. Furthermore, via numerical simulations, the model dynamics exhibits both Allee effect and diffusion controlled pattern formation growth to holes, stripe–hole mixtures, stripes, stripe–spot mixtures, and spots replication. That is to say, the dynamics of the model with Allee effect is not simple, but rich and complex.

1. Introduction

The understanding of patterns and mechanisms of spatial dispersal of interacting species is an issue of significant current interest in conservation biology and ecology, and biochemical reactions [1]. Interactions among species are multiform in communities [2,3]. Among these, a typical type of interactions is the one between a pair of predator and prey. The predator–prey model plays a major role in the studies of biological invasion of foreign species, epidemic spreading, extinction/spread of flame balls in combustion or autocatalytic chemical resection.

A prototypical predator–prey interaction model is of form [4,5]:

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} = a(u) - f(u)g(v), \\ \dfrac{\mathrm{d}v}{\mathrm{d}t} = \sigma f(u)g(v) - z(v), \end{cases} \tag{1}$$

where $u(t)$ and $v(t)$ are the densities of the prey and the predator at time $t > 0$, respectively.

In model (1), $a(u)$ is the growth rate of the prey in the absence of predation, which can be given by

$$a(u) = \alpha u \min\left\{1, \frac{K-u}{K-\varepsilon}\right\}, \quad \alpha > 0,\ \varepsilon \geq 0,\ K > 0,$$

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[*] Corresponding author. Tel.: +86 0577 86689222.
E-mail address: weimingwang2003@163.com (W. Wang).

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where, the parameter α stands for the specific growth rate of prey u, and K for the carrying capacity of the prey. If $\varepsilon = 0$, it reduces to the traditional logistic form $a(u) = \alpha u\left(1 - \frac{u}{K}\right)$.

The product $f(u)g(v)$ gives the rate at which prey is consumed. The prey consumed per predator, $f(u)g(v)/v$, was termed the functional response by Solomon [6]. These functions can be defined in different ways. In Refs. [4,5], Harrison defined

$$f(u) = \frac{\omega u}{\phi u + 1},$$

which is known as a Holling Type II functional response, and ω is the maximum rate of prey consumption by a single predator and ϕ is the half-saturation constant, the level of prey at which half the maximum consumption rate occurs. In this paper, following Lotka [7], we adopt

$$f(u) = cu, \tag{2}$$

which is a linear functional response without saturation, where $c > 0$ denotes the capture rate [8]. And following Harrison [4,5], we set:

$$g(v) = \frac{v}{mv + 1}, \tag{3}$$

where $m > 0$ represents a reduction in the predation rate at high predator densities due to mutual interference among the predators while searching for food. If $m = 0, g(v)$ reduces to the traditional form $g(v) = v$, indicating that the prey consumed is proportional to the number of predators, but there is evidence that there is mutual interference among predators searching for food, resulting in decreased consumption per predator as the predator density increases. This requires that $g(v)/v$ be decreasing, which can be modeled by using $\beta > 0$ [4,5]. Freedman [9] used $g(v) = v^m, 0 < m < 1$ to model predator mutual feeding interference, but Eq. (3) has the advantage that $v/(1 + mv) \approx v$ for small v, whereas $v^m/v \to \infty$ as $v \to 0$ [4,5].

The proportionality constant σ is the rate of prey consumption. And the function $z(v)$ is given by

$$z(v) = \gamma v + lv^2, \quad \gamma > 0, l \geq 0,$$

where γ denotes the natural death rate of the predator, and $l > 0$ can be used to model predator intraspecific competition that is not direct competition for food, such as some type of territoriality [4]. In this paper, we will discuss the case of $l = 0$, which is used in a much more traditional case.

Based on the above discussions, we can obtain the following model:

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} = \alpha u\left(1 - \dfrac{u}{K}\right) - \dfrac{cuv}{mv + 1}, \\ \dfrac{\mathrm{d}v}{\mathrm{d}t} = v\left(-\gamma + \dfrac{c\sigma u}{mv + 1}\right), \end{cases} \tag{4}$$

where α, K, γ have the same definitions as the above.

Let $\beta = \frac{\alpha}{K}$, $s = c\sigma$, model (4) leads to the following dimensionless equation:

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} = u\left(\alpha - \beta u\right) - \dfrac{cuv}{mv + 1}, \\ \dfrac{\mathrm{d}v}{\mathrm{d}t} = v\left(-\gamma + \dfrac{su}{mv + 1}\right), \end{cases} \tag{5}$$

where s denotes the conversion rate.

On the other hand, Allee effect in the population growth has been studied extensively. Allee effect, named after ecologist Warder Clyde Allee [10], is a phenomenon in biology characterized by a positive correlation between the population size or density and the mean individual fitness (oftentimes measured as per capita population growth rate) of a population or species [11], and may occur under several mechanisms, such as difficulties in finding mates when the population density is low, social dysfunction at small population sizes and increased predation risk due to failing flocking or schooling behavior, see Refs. [12–14]. In an ecological point of view, Allee effect has been divided into strong and weak cases. The strong Allee effect introduces a population threshold, and the population must surpass this threshold to grow. In contrast, a population with a weak Allee effect does not have a threshold. It has been attracting much more attention recently owing to its strong potential impact on the population dynamics of many plant and animal species [15]. Detailed investigations relating to Allee effect may be found in Refs. [16–25].

Allee effect has been modeled in different ways using various mathematical tools, and in most predation models it has been considered that Allee effect influences only the prey population. For instance, if $u = u(t)$ indicates the population size,