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Logarithmically improved blow-up criteria for the nematic liquid crystal flows[☆]

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ABSTRACT

We investigate blow-up criteria for the local in time classical solution of the nematic liquid crystal flows in dimensions two and three. More precisely, we prove that $0 < T_* < +\infty$ is the maximal time interval if and only if (i) for $n = 3$,

$$\int_0^{T_*} \frac{\|\omega\|_{\dot{B}^0_{\infty,\infty}} + \|\nabla d\|^2_{\dot{B}^0_{\infty,\infty}}}{\sqrt{1 + \ln(e + \|\omega\|_{\dot{B}^0_{\infty,\infty}} + \|\nabla d\|_{\dot{B}^0_{\infty,\infty}})}} \mathrm{d}t = \infty,$$

or

$$\int_0^{T_*} \frac{\|\nabla u\|^2_{\dot{B}^{-1}_{\infty,\infty}} + \|\nabla d\|^2_{\dot{B}^0_{\infty,\infty}}}{\ln(e + \|\nabla u\|_{\dot{B}^{-1}_{\infty,\infty}} + \|\nabla d\|_{\dot{B}^0_{\infty,\infty}})} \mathrm{d}t = \infty;$$

and (ii) for $n = 2$,

$$\int_0^{T_*} \frac{\|\nabla d\|^2_{\dot{B}^0_{\infty,\infty}}}{\ln(e + \|\nabla d\|_{\dot{B}^0_{\infty,\infty}})} \mathrm{d}t = \infty.$$

1. Introduction

Liquid crystals exhibit a phase of matter that has properties between those of a conventional liquid and those of a solid crystal, hence, it is commonly considered as the fourth state of matter, different from gases, liquid, and solid. Many different types of liquid crystal phases have been observed in practical experiments, distinguished by their characteristic optical properties such as birefringence. In the present state of knowledge, three main types of liquid crystals are distinguished, termed nematic, smectic and cholesteric. The nematic phase appears to be the most common, where the molecules do not exhibit any positional order, but they have long-range orientational order. Due to the physical importance and real-world applications, there have been numerous attempts to formulate continuum theories describing the behavior of liquid crystal flows, we refer to the seminal papers [1,2].

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In this paper, we are interested in the following hydrodynamic system modeling the flow of the nematic liquid crystal materials in n-dimensions ($n = 2$ or 3):

$$\partial_t u - \nu \Delta u + (u\cdot\nabla)u + \nabla P = -\lambda \nabla\cdot(\nabla d \odot \nabla d) \quad \text{in } \mathbb{R}^n \times (0,+\infty), \tag{1.1}$$

$$\partial_t d + (u\cdot\nabla)d = \gamma(\Delta d + |\nabla d|^2 d) \quad \text{in } \mathbb{R}^n \times (0,+\infty), \tag{1.2}$$

$$\nabla\cdot u = 0, \qquad |d| = 1, \quad \text{in } \mathbb{R}^n \times (0,+\infty), \tag{1.3}$$

$$(u,d)|_{t=0} = (u_0, d_0) \quad \text{in } \mathbb{R}^n, \tag{1.4}$$

where $u(x,t) : \mathbb{R}^n \times (0,+\infty) \to \mathbb{R}^n$ is the unknown velocity field of the flow, $P(x,t) : \mathbb{R}^n \times (0,+\infty) \to \mathbb{R}$ is the scalar pressure and $d(x,t) : \mathbb{R}^n \times (0,+\infty) \to \mathbb{S}^2$, the unit sphere in $\mathbb{R}^3$, is the unknown (averaged) macroscopic/continuum molecule orientation of the nematic liquid crystal flow, $\nabla\cdot u = 0$ represents the incompressible condition, u_0 is a given initial velocity with $\nabla\cdot u_0 = 0$ in the distribution sense, $d_0 : \mathbb{R}^n \to \mathbb{S}^2$ is a given initial liquid crystal orientation field, and ν, λ, γ are positive constants. The notation $\nabla d \odot \nabla d$ denotes the $n\times n$ matrix whose (i,j)-th entry is given by $\partial_i d \cdot \partial_j d$ $(1 \le i,j \le n)$. Since the concrete values of the constants ν, λ and γ play no role in our discussion, for this reason, we shall assume them to be all equal to one throughout this paper.

System (1.1)–(1.4) is a simplified version of the Ericksen–Leslie model (see [1,2]), which can be viewed as the incompressible Navier–Stokes equations (the case $d \equiv 1$, see [3–5]) coupling with the heat flow of a harmonic map (the case $u \equiv 0$, see [6,7]). Mathematical analysis of system (1.1)–(1.4) was initially studied by a series of papers by Lin [8] and Lin and Liu [9,10]. Later on, there were many extensive studies devoted to the nematic liquid crystal flows, see, e.g., [11–23] and the references therein. In the following, we briefly recall some related mathematical results of the liquid crystal flows. In paper [15], Lin, Lin and Wang established the global existence of Leray–Hopf type weak solutions to (1.1)–(1.4) on the bounded domain in $\mathbb{R}^2$ under suitable initial and boundary value conditions. When the space dimension $n \ge 3$, Li and Wang in [14] established the existence of a local strong solution with large initial value and the global strong solution with small initial value for the initial–boundary value problem of system (1.1)–(1.4). Wang in [7] proved that if the initial data $(u_0, d_0) \in BMO^{-1} \times BMO$ is sufficiently small, then there exists a global mild solution of system (1.1)–(1.4). Lin and Wang [16] obtained that when the initial data (u_0, d_0) satisfy $u_0, \nabla d_0 \in L^n(\mathbb{R}^n)$, the solution $(u,d) \in C([0,T]; L^n(\mathbb{R}^n)) \times C([0,T]; \dot{W}^{1,n}(\mathbb{R}^n, \mathbb{S}^2))$ to system (1.1)–(1.4) is unique.

In the present paper, we are interested in the short time classical solution of system (1.1)–(1.4). Since the strong solutions of the heat flow of harmonic maps must be blowing up at finite time (see [24]), we cannot expect that (1.1)–(1.4) has a global smooth solution with general initial data. It is well known that if the initial velocity $u_0 \in H^s(\mathbb{R}^n, \mathbb{R}^n)$ with $\nabla\cdot u_0 = 0$ and $d_0 \in H^{s+1}(\mathbb{R}^n, \mathbb{S}^2)$ for $s \ge n$, then there exists $0 < T_* < +\infty$ depending only on the initial value such that system (1.1)–(1.4) has a unique local classical solution $(u,d) \in \mathbb{R}^n \times [0, T_*)$ satisfying (see for example [22])

$$\begin{aligned} &u \in C([0,T]; H^s(\mathbb{R}^n, \mathbb{R}^n)) \cap C^1([0,T]; H^{s-2}(\mathbb{R}^n, \mathbb{R}^n)) \quad \text{and} \\ &d \in C([0,T]; H^{s+1}(\mathbb{R}^n, \mathbb{S}^2)) \cap C^1([0,T]; H^{s-1}(\mathbb{R}^n, \mathbb{S}^2)) \end{aligned} \tag{1.5}$$

for all $0 < T < T_*$. Here, we emphasize that such an existence theorem gives no indication as to whether solutions actually lose their regularity or the manner in which they may do so. Assume that such T_* is the maximum value for (1.5) holds, the purpose of this paper is to characterize such a T_*.

For the well-known Navier–Stokes equations with dimension $n \ge 3$, the Serrin conditions (see [25,5]) state that if $0 < T_* < \infty$ is the first finite singular time of the smooth solutions u, then u does not belong to the class $L^\alpha(0, T_*; L^\beta(\mathbb{R}^n))$ for all $\frac{2}{\alpha} + \frac{n}{\beta} \le 1$, $2 < \alpha < \infty$, $n < \beta < \infty$. Beale, Kato and Majda in [26] proved that the vorticity $\omega = \nabla \times u$ does not belong to $L^1(0, T_*; L^\infty(\mathbb{R}^n))$ if T_* is the first finite singular time. Later on, Kozono and Taniuchi [4], Kozono, Ogawa and Taniuchi [27] and Guo and Gala [28] improved the results of [26] into BMO and Besov space, more precisely, if T_* is the first singular time, then there hold

$$\int_0^{T_*} \|\omega\|_{BMO} \mathrm{d}t = +\infty;$$

$$\int_0^{T_*} \frac{\|\omega\|_{\dot{B}^0_{\infty,\infty}}}{\sqrt{1 + \ln(1 + \|\omega\|_{\dot{B}^0_{\infty,\infty}})}} \mathrm{d}x = +\infty,$$

where $\dot{B}^0_{\infty,\infty}$ denotes the homogeneous Besov space. We also refer the readers to [29] for more logarithmically improved criteria for the Navier–Stokes equations. On the other hand, as for the heat flow of harmonic maps into $\mathbb{S}^2$, Wang in [21] established that for $n \ge 2$, the condition $\nabla d \in L^\infty(0,T; L^n(\mathbb{R}^n))$ implies that the solution d is regular on $(0,T]$, i.e., $d \in C^\infty((0,T] \times \mathbb{R}^n)$. For the system (1.1)–(1.4), when dimension $n = 2$, Lin, Lin and Wang obtained that the local smooth solution (u,d) to (1.1)–(1.4) can be continued past any time $T > 0$ provided that there holds

$$\int_0^T \|\nabla d\|_{L^4}^4 \mathrm{d}t < +\infty.$$