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On the existence and uniqueness of limit cycles in planar continuous piecewise linear systems without symmetry

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HIGHLIGHTS

- Non-symmetric planar continuous piecewise-linear differential systems are studied.
- Some results about the existence and uniqueness of their limit cycles are given.
- For systems with three linear zones and no symmetries new results are obtained.
- For systems with two linear zones a shorter proof of known results is achieved.
- The application to the McKean model of a single neuron activity is described.

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ABSTRACT

Some techniques to show the existence and uniqueness of limit cycles, typically stated for smooth vector fields, are extended to continuous piecewise-linear differential systems.

New results are obtained for systems with three linearity zones without symmetry and having one equilibrium point in the central region. We also revisit the case of systems with only two linear zones giving shorter proofs of known results.

A relevant application to the McKean piecewise linear model of a single neuron activity is included.

1. Introduction and statement of main results

For planar differential systems, the analysis of the possible existence of limit cycles and their uniqueness is a problem which has attracted the interest of many works in the past. For smooth systems, good classical references in the field are the books [1,2]. The restriction of this problem to polynomial differential equations is the well-known 16th Hilbert's problem [3]. Since Hilbert's problem turns out to be a strongly difficult one, Smale [4] has particularized it to Liénard differential systems in his list of problems for the present century.

For just continuous or even smooth Liénard systems there are many results on the non-existence, existence and uniqueness of limit cycles, see for instance [5–9,2]. Going beyond the smooth case, the first natural step is to allow non-smoothness while keeping the continuity, as it has been done in some recent works [10–13]. In a further step, other authors have considered a line of discontinuity in the vector field defining the planar system, see [14,15].

In this paper, we adapt some techniques from the smooth case to continuous piecewise linear differential systems, obtaining new results for systems without symmetry. We also revisit the case of systems with only two linear zones giving shorter proofs of known results.

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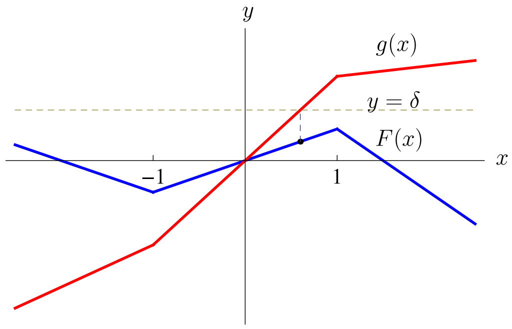

Fig. 1. The graphs of F and g considered in the paper have no symmetries. The point indicates the only equilibrium point, which initially is not at the origin.

In most interesting applications, continuous piecewise linear differential systems have two or three different linearity regions separated by parallel straight lines, see [16]. For such systems we assume without loss of generality that the lines separating these regions are $x = -1$ and $x = 1$. Furthermore, it is rather usual for these systems to exhibit only one anti-saddle singular point, that is one equilibrium point of focus or node type. Such a point is supposed to be in the central linearity region when the system has three linear zones, to be denoted in the sequel as L (left), C (central), and R (right). If the system has only two zones, we assume that the left and the central zones are in fact only one. As shown in [16], under these general assumptions, continuous piecewise linear differential systems (CPWL, for short) can be written in the Liénard form

$$\begin{aligned} \dot{x} &= F(x) - y, \\ \dot{y} &= g(x) - \delta, \end{aligned} \tag{1}$$

where

$$F(x) = \begin{cases} t_R(x-1) + t_C & \text{if } x \geq 1, \\ t_C x & \text{if } |x| \leq 1, \\ t_L(x+1) - t_C & \text{if } x \leq -1, \end{cases} \tag{2}$$

and

$$g(x) = \begin{cases} d_R(x-1) + d_C & \text{if } x \geq 1, \\ d_C x & \text{if } |x| \leq 1, \\ d_L(x+1) - d_C & \text{if } x \leq -1. \end{cases} \tag{3}$$

As mentioned before, we assume for the equilibrium point to be an anti-saddle in the band $-1 < x < 1$. This requires for the determinant in the central region to be positive, that is, $d_C > 0$, and also $-d_C < \delta < d_C$, along with $d_L, d_R \geq 0$. Thus, the only equilibrium point is located at the line $x = \bar{x} = \delta/d_C \in (-1, 1)$. The corresponding traces t_L, t_C, t_R could be arbitrary, but we know from the Bendixson theorem, see for instance Theorem 7.10 in [17], that they cannot have the same sign for the existence of limit cycles.

Remark 1. The above formulation includes as particular cases the following ones. If $t_C = t_L$ and $d_C = d_L$ then we have a system with only two different linearity zones, thoroughly analyzed in [11]. If $t_R = t_L$, $d_R = d_L$ and $\delta = 0$, then we have a symmetric system with three different linearity zones, first considered in [10] and thoroughly analyzed in [18]. The case $t_R = t_L$, $d_R = d_L$ and $\delta \neq 0$ was considered in [19]. Some relevant applications of this last situation have appeared in [20].

Under suitable hypotheses that include systems without any symmetry, see Fig. 1, our main results are the following.

Theorem 1. *Consider the differential system (1)–(3) with only one equilibrium point in the central zone, i.e.* $d_C > 0$, $-d_C < \delta < d_C$, *and* $d_L, d_R \geq 0$. *If the external traces satisfy* $t_L, t_R < 0$, *while the central trace is positive, that is* $t_C > 0$, *then the equilibrium point is surrounded by a limit cycle which is unique and stable.*

Theorem 1 is shown in Section 5, and of course, by using the opposite sign distribution for the traces, we could state a similar theorem on the existence and uniqueness of an unstable limit cycle.

For the case of only two linearity zones the corresponding result is a bit more involved.

Theorem 2. *Consider the differential system (1)–(3) with only two linearity zones, more specifically, under the assumptions* $t_C = t_L$ *and* $d_C = d_L > 0$, *and one equilibrium point in the left zone, i.e.* $\delta < d_C$, *and* $d_R \geq 0$. *Assume also that the left trace satisfies* $t_L > 0$, *while the right trace is negative, that is* $t_R < 0$. *Then the following statements hold.*

(a) *A necessary condition for the existence of periodic orbits is that the equilibrium point be a topological focus, that is* $t_L^2 - 4d_L < 0$.