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Asymptotic stabilization of coupled oscillators with dry friction by feedback control ☆

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ABSTRACT

In this paper we design a feedback control $u = u(x, \dot{x})$ so that each solution $x(\cdot)$ of the closed-loop system $\ddot{x}(t) + \partial\Phi(\dot{x}) + \nabla f(x) + u(x, \dot{x}) \ni 0$ approaches the set of critical points of $f(x)$ with $|\dot{x}(t)| \to 0$ as $t \to +\infty$. The robustness of the control is also discussed in the case where $f(x)$ has only a finite number of critical values. The approach is mainly based on LaSalle's invariance principles and Morse decomposition theory of attractors for differential inclusions.

1. Introduction

This paper is concerned with the asymptotic stabilization of the following nonsmooth mechanical system

$$\ddot{x} + \partial\Phi(\dot{x}) + \nabla f(x) \ni 0, \quad t \geq 0, \tag{1.1}$$

where $x \in X := \mathbb{R}^m$, $f \in C^1(X)$ is a smooth potential, and

$$\Phi(\dot{x}) = a|\dot{x}| + \Psi(\dot{x}), \quad a \geq 0, \tag{1.2}$$

where $\Psi : X \to \mathbb{R}^1$ is a C^1 convex function with $\nabla\Psi(0) = 0$. Such systems are widely used to describe coupled oscillators with dry friction in which each particle is connected to its neighbors by harmonic springs [1]. (They can also be derived from the spatial discretization of a vibrating string equation subject to solid friction [2].) Indeed, if we take $\Phi(y) = |y|$, then (1.1) reduces to the mathematical model describing oscillators with a Coulomb friction, while in the case where $\Phi(y) = |y|^2$ it gives a viscous one and the associated "Heavy Ball with Friction" system [3]. Notice also that the function $\Phi(y) = |y|^p$ $(p \in (1, 2))$ generates an intermediate situation, which was considered by Amann–DÍaz [4] and DÍaz and Liñán [5] under the terminology of "strong friction ". In the general case the system provides us with a unifying framework whose asymptotic behavior was extensively investigated in the recent works of Adly and Goeleven [6], Cabot [2], etc.

In general the dynamical behavior of nonsmooth systems seems to be more complicated than that of smooth ones. We observe that in (1.1), even if the potential $f(x)$ is a smooth function which has only a finite number of critical points, the nonsmooth system may have uncountably many equilibrium points. Indeed, it can be easily seen that any $x_\infty \in \mathbb{R}^m$ with $-\nabla f(x_\infty) \in \partial\Phi(0)$ is an equilibrium of the system. It was also shown that each unique solution of the system converges in a finite time toward an equilibrium state x_∞ provided that $-\nabla f(x_\infty) \in \operatorname{int} \partial\Phi(0)$ [7]. Such a phenomena never occurs in a smooth dynamical system.

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In practice one naturally hopes that a nonsmooth system could behave very much like a smooth one. However, in the situation of uncontrolled systems this can be hardly fulfilled. In this short note we add a control u into the system (1.1), namely, we consider the controlled system

$$\ddot{x}(t) + \partial\Phi(\dot{x}) + \nabla f(x) + u \ni 0, \quad t \geq 0. \tag{1.3}$$

Our main purpose is to design a feedback law so that the long time dynamics of the system is much simpler then that of the uncontrolled one. Specifically, we will formulate explicitly a function $u = u(x, \dot{x})$ so that each solution $x(\cdot)$ of the closed-loop system

$$\ddot{x}(t) + \partial\Phi(\dot{x}) + \nabla f(x) + u(x, \dot{x}) \ni 0, \quad t \geq 0 \tag{1.4}$$

approaches the set M of critical points of $f(x)$ along with $|\dot{x}(t)| \to 0$ as $t \to +\infty$, where

$$M = \{z | \nabla f(z) = 0\}.$$

We also show that such a feedback control is robust with respect to small perturbations and measurement errors etc. in case f has only a finite number of critical values. The approach here is mainly based on LaSalle's invariant principles and Morse decomposition theory of strong attractors for differential inclusions.

This paper is organized as follows. In Section 2 we make some preliminary works, and in Section 3 we recall a special case of LaSalle's invariance principles for differential inclusions and discuss the robustness of strong LaSalle sets. Section 4 is devoted to the design of the feedback control and its robustness property for (1.3).

2. Some dynamical concepts on differential inclusions

In this section we will recall some basic definitions and results on the dynamics of differential inclusions.

Let $X = \mathbb{R}^m$. For any $x, y \in X$, we will use $x \cdot y$ (or sometimes $\langle x, y\rangle$) and $|x|$ to denote the usual inner product of x and y and the norm of x, respectively.

Let M be a subset of X. The *closure* and *interior* of M are denoted by $\overline{M}$ and int$M$, respectively. We say that a subset V of X is a *neighborhood* of M, this means $\overline{M} \subset \text{int}V$. We will use $\mathcal{B}(A, r)$ to denote the *r-neighborhood* of A, i.e.,

$$\mathcal{B}(M, r) = \{y \in X \mid d(y, M) < r\}.$$

Consider the differential inclusion

$$\dot{x}(t) \in F(x(t)), \quad x(t) \in \Omega, \tag{2.1}$$

where Ω is an open subset of $\mathbb{R}^m$, and F is a multifunction on Ω.

Definition 2.1. Let I be an interval. A map $x(\cdot) : I \to X$ is said to be a *solution* of (2.1) on I, if it is absolutely continuous on any compact interval $J \subset I$ and solves (2.1) at a.e. $t \in I$.

A solution $x(\cdot)$ of (2.1) on $\mathbb{R}^1$ will be simply called an *entire solution*.

From now on we will always assume that (2.1) satisfies the following *Standing Hypotheses* (H):

(H1) $F(x)$ is a nonempty compact convex subset of X for each $x \in \Omega$;
(H2) The multifunction F is upper semicontinuous.

It is well known that this guarantees local existence of solutions, that is, for every $x \in \Omega$ there exists a solution $x(\cdot)$ of (2.1) satisfying $x(0) = x$ on some maximal interval $[0, T_x)$.

For $x \in \Omega$, we denote by $\mathcal{S}_F(x)$ the family of solutions $x(\cdot)$ of (2.1) with initial value $x(0) = x$. The *reachable mapping* $\mathcal{R}(t)$ of (2.1) is defined as follows:

$$\mathcal{R}(t)x = \{x(t) \mid x(\cdot) \in \mathcal{S}_F(x)\}, \quad \forall\, (t, x) \in \mathbb{R}^+ \times X.$$

It is well known that $\mathcal{R}(t)$ possesses semigroup properties.

Let M, N be two subsets of Ω. We say that M *attracts* N, this means that no solution $x(\cdot)$ with $x(0) \in N$ blows up in finite time, moreover, for any $\varepsilon > 0$ there exists a $T > 0$ such that

$$\mathcal{R}(t)N \subset \mathcal{B}(M, \varepsilon), \quad t > T.$$

The *attraction basin* $\mathrm{B}(M)$ of M is defined as

$$\mathrm{B}(M) = \{x \in \mathbb{R}^m \mid M \text{ attracts } x\}.$$

M is said to be *strongly positively invariant* (resp. *invariant*), if

$$\mathcal{R}(t)M \subset M \text{ (resp. } \mathcal{R}(t)M = M), \quad \forall t \geq 0.$$

M is said to be *weakly invariant*, if for any $x \in M$ there passes through x an entire solution $x(\cdot)$ with $x(t) \in M$ for all $t \in \mathbb{R}^1$.