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Effect of cross-diffusion on the stationary problem of a diffusive competition model with a protection zone

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ABSTRACT

This paper is concerned with the positive stationary problem of a Lotka–Volterra cross-diffusive competition model with a protection zone for the weak competitor. The detailed structure of positive stationary solutions for small birth rates and large cross-diffusion is shown. The structure is quite different from that without cross-diffusion, from which we can see that large cross-diffusion has a beneficial effect for the existence of positive stationary solutions. The effect of the spatial heterogeneity caused by protection zones is also examined and is shown to change the shape of the bifurcation curve. Thus the environmental heterogeneity, together with large cross-diffusion, can produce much more complicated stationary patterns. Finally, the asymptotic behavior of positive stationary solutions for any birth rate as the cross-diffusion coefficient tends to infinity is given, and moreover, the structure of positive solutions of the limiting system is analyzed. The result of asymptotic behavior also reveals different phenomena from that of the homogeneous case without protection zones.

1. Introduction

Since the fact that ecosystems are fundamentally influenced by spatial heterogeneity has been observed in many scientific experiments [1], more and more interesting papers investigating the effects of the environmental heterogeneity have appeared in recent years. As pointed out in [2] or [3], it is not easy to capture the influence of heterogeneous spatial environment on population models in general. To account for the environmental heterogeneity, models with coefficients of functions of the space variable *x* are considered instead of those with constant coefficients. But the mathematical techniques developed to study these models are typically either not sensitive to this change, such as the bifurcation approach, the topological degree approach and the upper and lower solution argument, in which case the effects of heterogeneous spatial environment are difficult to observe in the mathematical analysis, or too sensitive to this change, such as the various Lyapunov function techniques, and become inapplicable in the heterogeneous case.

Observing that the behavior of these models are quite sensitive to certain coefficient functions becoming small in part of the underlying spatial region, Du et al. [4–7,2,8] have successfully used such degeneracy, i.e. the self crowding effect for one of the species is assumed to be zero or small on part of the habitat, to capture the effects of spatial heterogeneity on the Lotka–Volterra competition model and prey–predator models, and reveal essential differences of the models' behaviors from the homogeneous case. Hutson et al. [9–13] have investigated spatial effects of birth rates in some diffusive competition models. In particular, by perturbing the limiting case where the two species are identical, some interesting phenomena are revealed in [14] by Lou et al. and [9,12]. In addition to the above work, the effect of environmental change caused by the creation of a simple protection zone has been studied for Lotka–Volterra competition models and prey–predator models

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in [15–17]. One can also refer to a book by Cantrell and Cosner [18, Chapter 6] and references therein for effects of the spatial heterogeneity. We point out that the mentioning work is devoted to the reaction–diffusion models, and the effect of cross-diffusion is not considered.

The effect of cross-diffusion has been studied in many papers, e.g. it can create interesting pattern formation; one can see [19–25] and references therein for more effects about cross-diffusion. Then it is an interesting problem to study the combined effects of spatial heterogeneity and cross-diffusion on population models. To the best of our knowledge, Oeda [26] investigated a cross-diffusive prey–predator system with a protection zone for the prey, and showed the effects of cross-diffusion on positive stationary solutions.

In this paper, we intend to study the effects of environmental heterogeneity with large cross-diffusion in the following Lotka–Volterra cross-diffusive competition model with a protection zone:

$$\begin{cases} u_t = \Delta\left[(1+k\rho(x)v)\,u\right] + u\,(\lambda - u - \eta b(x)v)\,, \quad x\in\Omega,\, t>0,\\ \tau v_t = \Delta v + v\,(\mu - v - du)\,, \qquad x\in\Omega\setminus\bar{\Omega}_0, \quad t>0,\\ \partial_\nu u = 0, \quad x\in\partial\Omega,\, t>0, \qquad \partial_\nu v=0, \qquad x\in\partial\Omega\cup\partial\Omega_0, \quad t>0,\\ u(x,0)=u_0(x)\ge 0, \quad x\in\bar{\Omega}, \qquad v(x,0)=v_0(x)\ge 0, \quad x\in\bar{\Omega}\setminus\bar{\Omega}_0. \end{cases} \tag{1.1}$$

Here $\Omega \subseteq \mathbb{R}^N$ $(N\ge 1)$ is a bounded domain with smooth boundary $\partial\Omega$, Ω_0 is a subdomain of Ω with smooth boundary $\partial\Omega_0$ and $\bar{\Omega}_0\subset\Omega$; ν is the outward unit normal vector on the boundary and $\partial_\nu = \partial/\partial\nu$; positive constants λ and μ are the intrinsic growth rates of the respective species; $\eta b(x)$ and $d>0$ are the inter-specific competitive pressure on u and v, respectively; $\rho(x)$ and $b(x)$ are spatially heterogeneous, and satisfies $\rho(x)\equiv b(x)\equiv 0$ in $\bar{\Omega}_0$ and $\rho(x)\equiv 1>0$ and $b(x)\equiv b>0$ in $\bar{\Omega}\setminus\bar{\Omega}_0$; $\tau>0$ is a constant; $u(x,t)$ and $v(x,t)$ represent the population densities of the respective competing species. In the model, u lives in the larger habitat Ω, and Ω_0 is its protection zone, where u can leave and enter the protection zone freely, while v can only live outside Ω_0. Thus we impose a no-flux boundary condition on $\partial\Omega_0$ for v. On $\partial\Omega$, a no-flux boundary condition is also assumed for both species, and no individuals cross the boundary $\partial\Omega$. Throughout the paper, we write $\Omega_1=\Omega\setminus\bar{\Omega}_0$.

When there is no protection zone $\Omega_0=\emptyset$, we know that if $\lambda > b\mu\eta$, $(0,\mu)$ is unstable; while if $0<\lambda<b\mu\eta$, $(0,\mu)$ is asymptotically stable. Then as η is large enough such that $\lambda<b\mu\eta$ holds, $(0,\mu)$ is asymptotically stable. One sees that the competitor u under rather strong competition pressure would be highly vulnerable to extinct without protection zones. Thus, to save the weak competitor u, it is natural to set up one or several protection zones for u. Then it is important to understand the effects of such protection zones on the affected species. In this paper, we investigate system (1.1) with a single simple protection zone for simplicity and show the effects of such heterogeneous environment under large cross-diffusion.

It should be noted that $k\Delta[\rho(x)vu]$ here is the cross-diffusion term which was originally proposed by Shigesada et al. [27] to model the habitat segregation phenomena between two competing species. From the cross-diffusion term, we know that u diffuses to low density regions of v in their common living habitat Ω_1, and the coefficient k denotes the sensitivity of the competitor u to the population pressure from the other competitor v. One can see the monograph by Okubo and Levin [28] for more ecological background.

Due to the presence of cross-diffusion, we can no longer treat system (1.1) as the reaction–diffusion systems without cross-diffusion by the powerful theory of monotone dynamical systems, and fail to obtain results as complete as those of the reaction–diffusion systems due to the limited mathematical technology. Instead, we attempt to show more details about the structure of the positive stationary solution set of (1.1) and the techniques used here are rather different from those mentioned above.

It is clear that the corresponding stationary problem is

$$\begin{cases} \Delta\left[(1+k\rho(x)v)\,u\right] + u\,(\lambda-u-\eta b(x)v) = 0, \quad x\in\Omega,\\ \Delta v + v\,(\mu - v - du) = 0, \quad x\in\Omega_1,\\ \partial_\nu u = 0, \quad x\in\partial\Omega, \qquad \partial_\nu v = 0, \quad x\in\partial\Omega_1. \end{cases} \tag{1.2}$$

From the ecological viewpoint, we are only interested in positive solutions of (1.2). It is said that (u,v) is a positive solution of (1.2) if $u>0$ in $\bar{\Omega}$ and $v>0$ in $\bar{\Omega}_1$, and a positive solution (u,v) corresponds to a coexistence state of the interacting species u and v. To show the structure of positive solutions of (1.2), we mainly combine the method of the bifurcation theory with the Lyapunov–Schmidt reduction, which were used by Du and Lou [29] in 1998 to obtain an S-shaped global bifurcation branch for a prey–predator system with Holling–Tanner response. In 2004, Kuto and Yamada [30] applied the methods to a Lotka–Volterra prey–predator system with cross-diffusion and also obtained an S-shaped global bifurcation branch. Recently, Kuto [31] has successfully used the methods to a Lotka–Volterra cross-diffusion prey–predator system in heterogeneous environment, and found that a spatial segregation of certain coefficients could change the shape of the bifurcation curve. In this paper, we expect to apply the methods to show the more detailed structure of the positive solution set of (1.2) for small λ and large cross-diffusion k. Furthermore, the asymptotic behavior of positive solutions of (1.2) as $k\to\infty$ and the structure of the positive solution set of the limiting system will also be shown, which provide important information on the structure of positive solutions of (1.2) for any $\lambda>0$ as the cross-diffusion k is large.