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On a vegetation pattern formation model governed by a nonlinear parabolic system

Kaloyan Marinov [a], Tianze Wang [b,c], Yisong Yang [c,d,*]

[a] Department of Mathematics, University of Washington, Seattle, WA 98195, USA

[b] School of Mathematics and Information Science, North China University of Water Resources and Electric Power, Zhengzhou 450011, Henan, PR China

[c] Institute of Contemporary Mathematics, School of Mathematics, Henan University, Kaifeng 475000, Henan, PR China

[d] Department of Mathematics, Polytechnic Institute of New York University, Brooklyn, NY 11201, USA

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ABSTRACT

A fundamental subject in ecology is to understand how an ecosystem responds to its environmental changes. The purpose of this paper is to study the desertification and vegetation pattern formation phenomena and understand the dependence of the biomass density B of vegetation on the level of available environmental water resources, controlled by a water supply rate parameter R, which is governed by a coupled system of nonlinear parabolic equations in a mathematical model proposed recently by Shnerb, Sarah, Lavee, and Solomon. It is shown that, when R is below the death rate μ of the vegetation in the absence of water, the solution evolving from any initial state approaches exponentially fast the desert state characterized by $B = 0$; when R is above μ, the solution evolves into a green vegetation state characterized by $B \not\to 0$ as time $t \to \infty$. In the flower-pot limit where the system becomes a system of ordinary differential equations, it is shown that nontrivial periodic vegetation states exist provided that the water supply rate R is a periodic function and maintains a suitable average level. Furthermore, some conservation laws relating the asymptotic values of the vegetation biomass B and available water density W are also obtained.

1. Introduction

It is well known that mathematical modeling [1,2] plays an important role in the quantitative and qualitative understanding of vegetation pattern formation and, in particular, the disappearance of vegetation, often referred to as desertification or desertization, when there is an adverse change in the living environment of the vegetation under consideration. Such a change may mainly be described in terms of one or several biogeochemical cycles including those of water, carbon, and nitrogen. In return, vegetation also regulates those biogeochemical cycles given as environmental resource parameters, thus, resulting in a highly coupled dynamical ecosystem governing the interaction of the vegetation biomass and the environmental resource parameters. How ecosystems respond to environmental resource changes is one of the main frontiers in ecology [3]. Although environmental resource change may be a slow and gradual process, an ecosystem may undergo a catastrophic change attributed to the existence of two alternative stable or attracting states generically referred to as the onset of bistability [3] in the ecosystem. Mathematical modeling aims at pursuing predicative power for these catastrophic responses to changing environments [3–10].

* Corresponding author.
E-mail address: yisongyang@gmail.com (Y. Yang).

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Recently, Shnerb et al. [11] proposed a general and simple model describing an interacting water–shrubs ecosystem that illustrates the expected bistable phase transition in terms of desertification and vegetation pattern formation picture with respect to the water resource change controlled by a water supply rate parameter. In their model, the density of shrubs biomass, B, and the available water density, W, are governed by a system of nonlinear parabolic equations for which the evolution of B and W are related to the death rate, μ, of the vegetation in the absence of water, and the water supply rate parameter, R. Shnerb et al. [11] obtained numerical results showing vegetation pattern formation in their model when R is sufficiently large and described linear stability of the two equilibrium solutions representing the desert and vegetation states.

In this paper, we present a mathematical analysis of the desertification and vegetation pattern formation model of Shnerb et al. [11]. One of our main results establishes that the *global* phase transition point between the desert state characterized by $B \to 0$ as $t \to \infty$ and the green vegetation state characterized by $B \not\to 0$ as $t \to \infty$ is sharply and explicitly rendered at the threshold level $R = R_c = \mu$. In other words, whether a solution starting from any initial state approaches the desert state is dictated by whether the explicit meager-water-supply condition $R < \mu$ holds true.

An outline of the rest of the paper is as follows. In the next section, we review the model of Shnerb et al. [11] and establish a three-part theorem regarding its solution. The first part concerns the well-posedness of the model. That is, we prove the global existence, uniqueness, and positivity of a solution to the governing system of equations of [11]. The second part concerns the occurrence of the desert state. We show that the desert state characterized by a vanishing biomass density, $B = 0$, is globally stable when $R < \mu$. The third part concerns the occurrence of a vegetation state. We consider the situation when $R > \mu$ and prove the occurrence of a vegetation state characterized by $\limsup_{t\to\infty}\{B\} = B_\infty > 0$. In Section 3, we present a complete analysis of the homogeneous or zero-dimensional limit of the model describing for example a "flower-pot" (in the words of [11]) situation. More precisely, based on the Poincaré–Bendixon theorem, we will prove that the steady green vegetation state, which is the unique stable equilibrium point of the governing ordinary differential equations, is globally stable, meaning that a solution starting from any nontrivial initial state evolves into the steady green vegetation state, $B \to B_v > 0$, exponentially fast as time $t \to \infty$. Besides, we also obtain an explicit formula for all solutions when $\mu = 1$, which gives useful information about what happens in the critical situation, $R = \mu$. Our results for the vegetation states, combined with the global stability result for desert state obtained in Section 2, implies that the bistability transition point expressed in terms of the water supply rate R is indeed rendered at the critical value $R = R_c = \mu$. In Section 4, we consider the situation when R is a periodic function of t. We prove by a fixed-point theory argument that the homogeneous system allows the existence of a periodic solution when the average value of R over its period is adequately given. Such a result is naturally expected for the onset of a "seasonal" response of the vegetation biomass and available water resource with respect to the change of water supply rate. In Section 5, we consider the full spatially dependent system of equations and study the asymptotic behavior of the solutions under the high-water-supply condition $R > \mu$. We will derive some uniform bounds for B and W and obtain some pointwise and asymptotic "conservation laws" relating or confining B and W. A detailed explanation of these relations will also be seen there. In Section 6, we extend our study to the generalized modeling equations considered in [11,12] where the vegetation death rate is taken to be a bounded positive-valued function of the vegetation biomass. We show that, depending on some specific conditions stated in terms of the coupling parameters, the system may allow the desert state and vegetation state to be globally stable, respectively, or allow the occurrence of periodic oscillations when the vegetation state loses its local stability. In Section 7, we summarize our results and state some comments.

2. Characterization of desert and vegetation states

Consider the interaction of water resource and vegetation biomass in an ecosystem so that the presence of water supply enhances vegetation and the latter, in turn, consumes the former. The evolution equations proposed by Shnerb et al. [11] governing such an interaction are given in the form

$$\frac{\partial B}{\partial t} = WB - \mu B, \tag{2.1}$$

$$\frac{\partial W}{\partial t} = D\Delta W + \mathbf{V}\cdot\nabla W + R - W - \lambda WB, \tag{2.2}$$

$$B(x, 0) = B_0(x), \qquad W(x, 0) = W_0(x), \tag{2.3}$$

where B is the density of shrubs biomass and W is the available water density, the parameters μ, λ, D, R are positive constants so that μ is the vegetation death rate, λ the water consumption rate in the presence of vegetation, D a diffusion constant taking account of water loss due to evaporation (say), and R a water rate due to rainfall (say). The initial states $B_0(x), W_0(x)$ in (2.3) are nonnegative functions of the spatial variable x, the term $\mathbf{V}\cdot\nabla W$ takes account of the downhill water loss in which $\mathbf{V}$ is a constant "terrain slope" vector, and the initial value problem consisting of Eqs. (2.1)–(2.3) are considered for time $t > 0$ and subject to a spatially periodic boundary condition over a planar spatial domain Ω, which may be viewed as a flat (closed) torus. For convenience, we assume that B_0 and W_0 are all smooth.

For greater generality, we often assume R to be a bounded function of t unless otherwise stated. We define the quantities

$$W^0 = \max\{W_0(x) \mid x \in \Omega\}, \qquad R^0 = \sup\{R(t) \mid t \geq 0\}, \qquad K = \max\{W^0, R^0\}. \tag{2.4}$$