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Weak Allee effect in a predator–prey model involving memory with a hump

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ABSTRACT

In this paper we consider a predator–prey system which has a factor that allows for a reduction in fitness due to declining population sizes, often termed an *Allee effect*. We study the influence of the *weak Allee effect* which is included in the prey equation and we determine conditions for the occurrence of Hopf bifurcation. The prey population is limited by the carrying capacity of the environment, and the predator growth rate depends on past quantities of the prey which is represented by a *weight function* that specifies a moment in the past when the quantity of food is the most important from the point of view of the present growth of the predator. The stability properties of the system and the biological issues of the memory and Allee effect on the coexistence of the two species are studied. Finally we present some simulations to verify the veracity of the analytical conclusions.

1. Introduction

In 1931, Allee (see [1]) had proposed that intraspecific cooperation might lead to inverse density dependence. Allee observed that many animal and plant species suffer a decrease of the *per capita* rate of increase as their populations reach small sizes or low densities. Under such conditions, the rate of increase can reach zero, or even negative values, because of a decrease in reproduction or when conspecific individuals are not numerous enough for survival. "Undercrowding, as well as overcrowding, may be limiting these population". In other words, the Allee effect refers to a population that has a maximal *per capita* growth rate at intermediate density. This occurs when the *per capita* growth rate increases with the increase in the density and decreases when the density passes through a critical value, which enhances their likelihood of extinction. Clearly this situation is different from the logistic growth in which the *per capita* growth rate is a decreasing function of the density.

As is known, in nature some species co-operate among themselves in their search for food or when they try to escape from predators. Allee [1,2], had studied extensively the aspects of aggregation and associated co-operative and social characteristics among the members of a species.

The Allee effect will be weak if there exists no critical density population, below which the *per capita* rate becomes negative. Taking into account the carrying capacity of the environment with respect to the prey in the *per capita* growth rate of the population, the weak Allee effect has been modeled in [3] by the following differential equation

$$\dot{N}(t) = \frac{\varepsilon}{K} N^2(t)\left(1 - \frac{N(t)}{K}\right) \tag{1}$$

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in which $N(t)$ denotes the prey population at time t, $0 < \varepsilon < 1$ is the *per capita* growth rate of the population and $K > 0$ is the carrying capacity of the environment.

It is natural to introduce time delay into models of interacting species, formulated in terms of coupled ordinary differential equations. For example, in predator–prey models it is reasonable to assume that the rate of increase of the predator population depends on the recent history of the prey population. So, in this study we propose and analyze the following system of equations

$$\begin{aligned} \dot{N} &= \frac{4\varepsilon}{K} N^2 \left(1 - \frac{N}{K}\right) - \alpha NP \\ \dot{P} &= -\gamma P + \beta P \int_{-\infty}^{t} N(\tau)\, G(t-\tau)\, d\tau, \end{aligned} \tag{2}$$

that describe the dynamics of interactions between a predator and a prey species with a weak Allee effect in the prey populations $N(t)$ and a distributed delay $G(t-\tau)$ in the predator population $P(t)$, at time t. The parameters $K, \varepsilon, \alpha, \gamma$ and β are positive. K represents the carrying capacity of the environment with respect to the prey, ε is the intrinsic growth rate of the prey, α is the rate of predation, γ is the intrinsic mortality of the predator and β is the conversion rate. In model (2) we assume that the per capita growth rate of the prey varies, according to space and environmental resources available, as follows: when the population is small the per capita growth rate increases until reaching its maximum value at $N = \frac{K}{2}$ and from that point starts to decrease until it reaches the carrying capacity of the system. This behavior generates a magnification of ε which is denoted by 4ε in the model (2). This magnification renders clarity to the simulations carried in this paper.

The distributed delay represented by the weight function $G : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies $G(t) \geq 0$ and $\int_0^\infty G(t)dt = 1$. See [4]. The continuous delay in the predator equation means that the quantity of prey has an influence on the present growth rate of predator not just at a single moment in the past but over the whole past, or at least in those time intervals where the function $G(t)$ is not zero. Our study in this paper treats the case of the density function $G : \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$G(t) = a^2 t e^{-at}, \quad a > 0, \tag{3}$$

which is termed as ("*memory with a hump*") and is a particular case of the Gamma Function described by Fargue [5].

The models considered in earlier studies [6–9,4,10,11] were similar, but our model differs from those in terms of incorporating weak Allee effect as opposed to a logistic functional response in the prey dynamics. In recent studies [12–18] the authors have considered in continuous models of differential equations the cases of strong and weak Allee effect and fulfilled a comprehensive bifurcation analysis to study the existence of periodic solutions and their stability properties. In [19] the authors have considered a discrete-time competition model where each species suffers from either predator saturation induced Allee effects and/or mate limitation induced Allee effects and analyze the coexistence between the species. The authors in [20] have characterized the bifurcation of the nontrivial equilibrium of the system (2) under the assumption that the function $G(t)$ is defined by

$$G(t) = a e^{-at}, \quad a > 0 \tag{4}$$

that is the weight function is exponentially decaying ("*exponentially fading memory*").

If the weight function in model (2) is given by (4), the effect of the past is fading away exponentially as we go backwards in time. There is a *critical delay* under which the bifurcation is supercritical, and above which it is subcritical. On the other hand, if the weight function is given by (3), we have the advantage that it specifies the moment in the past when the quantity of food is the most important from the point of view of the present growth of the predator. This occurs $\frac{1}{a}$ units before the present time t (the weight function has a hump at $\tau = t - \frac{1}{a}$, and going further backwards in time the effect of the past is fading away), the phenomenon is richer. In this case we have to take also into account the value ε of the intrinsic growth rate of prey. For small fixed ε the bifurcation is always supercritical for arbitrary large delay, i.e. after the loss of stability of the equilibrium the system exhibits small amplitude stable oscillations. For large fixed ε the bifurcation is supercritical for small and for very large delays, however, there is an interval of possible delays such that if the delay falls into this interval the bifurcation is subcritical.

In view of the consideration given in the previous paragraph, we see that the model (2) describing the dynamical behavior of the predator–prey system where $G(t)$ is given by (3), differs considerably from that studied in [20] where $G(t)$ is given by (4) although we may get analogous stability results for both. Furthermore, the analysis is more difficult and requires the use of sophisticated algorithmic procedures to write the expressions of the Lyapunov coefficients to show the occurrence of Hopf bifurcation in the system. In Section 3 we present a brief summary of the technique to be used to calculate the first Lyapunov coefficient which will be crucial to determine the direction of the Hopf bifurcation.

It is well known in biology that group defense helps decrease (or even prevent) the predation due to the enrichment in the ability of the prey to defend or escape from the predators. In view of the considerations in [21], the model Eq. (2) may also be viewed as a predator–prey system with group defense exhibited by the prey. It is observed that model (2) exhibits interesting dynamical behavior of the interacting populations due to the presence of the weak Allee effect different from