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Topological bifurcations of central configurations in the N-body problem

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ABSTRACT

In this article we study topological bifurcations of classes of central configurations of the spatial 6- and 7-body problems. We treat these classes as $SO(3)$-orbits of critical points of a family of $SO(3)$-invariant potentials. Using the equivariant bifurcation theory technique, we prove the existence of a global topological bifurcation of classes of central configurations in the 7-body problem and a local topological bifurcation in the 6-body problem.

1. Introduction

The central configurations play a crucial role in the study of the Newtonian N-body problem of celestial mechanics, one of the main applications is the fact that they allow us to obtain the unique explicit solutions known until now in this classical problem, called homographic solutions, for which the configuration of the particles is central for all time. This was pointed out first by Euler and Lagrange; historically the problem of central configurations was first formulated in this context. For a classical background on this subject we refer the reader to [1,2], while a more recent work can be found in [3,4]. Another important role of the central configurations comes from the fact that the total collision or the total parabolic escape to infinity in the N-body problem is always asymptotic to a central configuration. If we fix the well known first integrals, the total energy h and the angular momentum c, the topology of the invariant manifolds with h and c constant changes precisely at the central configurations. Central configurations have also been studied in models where the particles move following forces different from the Newtonian ones, see for instance [5,6].

The aim of this paper is to study classes of spatial central configurations of the N-body problem as $SO(3)$-orbits of critical points of families of $SO(3)$-invariant potentials, where $SO(3)$ is the group of three-dimensional orthogonal matrices of determinant 1. The basic idea is to apply the equivariant bifurcation theory. We study topological bifurcations of some families of central configurations in the $(n+3)$-body problem, for $n = 3, 4$. More precisely, we make use of the equivariant Conley index, see [7–9], to obtain a local topological bifurcation of orbits of critical points and the degree for equivariant gradient maps, [10–12,9,13], to obtain a global topological bifurcation of such orbits. This kind of bifurcations has been widely studied in the non-equivariant case, see for instance [8,14–18].

After the introduction, our paper is organized as follows: in Section 2 we give the preliminaries for the analysis of the existence and bifurcations of classes of spatial central configurations of the N-body problem. We start this section with the formal definition of a central configuration of the N-body problem, see Definition 2.1. Recently, in a nice article, Fernandes and Mello have proved the existence of some new families of classes of central configurations, see [19]. Two of these families, denoted in this article by (6) and (10), are the starting point of our main results whose statement appears in Theorems 2.1 and 2.2. In these theorems we give necessary and sufficient conditions for the existence of a global and a local bifurcation

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of classes of central configurations from families (6) and (10). We treat classes of central configurations of the spatial N-body problem as $SO(3)$-orbits of solutions of Eqs. (5) and (9) i.e. as $SO(3)$-orbits of critical points of families of $SO(3)$-invariant potentials (4) and (8). We recall that the bifurcation phenomenon can occur only at degenerate $SO(3)$-orbits, see also [4]. These theorems describe degenerate $SO(3)$-orbits of Fernandes and Mello families (6) and (10), respectively. To find degenerate $SO(3)$-orbits we study the Hessian of potentials (4) and (8) along families (6) and (10), respectively. Theorem 2.1 ensures the existence of a global bifurcation of $SO(3)$-orbits from the family (6), it states that from all degenerate classes of central configurations of the family (6) bifurcate connected sets of classes of central configurations of the 7-body problem. Theorem 2.2 yields information about local bifurcations of classes of central configurations from family (10). Namely, this theorem states that any degenerate class of central configurations of the family (10) is not isolated in the set of non-trivial classes.

The abstract theory presented in the last section can be applied to the study of classes of central configurations of N-body problems for an arbitrary N. Our viewpoint sheds some new light on topological bifurcations of classes of central configurations of the N-body problem. We have chosen the families (6) and (10) just to illustrate our approach. In our subsequent papers we are going to continue the study of the existence, continuation and bifurcations of classes of central configurations of the N-body problem applying the critical point theory in the presence of symmetries of a compact Lie group.

The proofs of our main results appear in Section 3.

In Section 4 we have formulated some remarks, open questions and conjectures concerning results presented in this article.

In order to have a self-contained paper, we have included the Appendix, where we review some of the standard facts on equivariant bifurcation theory. We emphasize that critical points of invariant potentials usually are not isolated. The advantage of using bifurcation theory in the presence of symmetries of a compact Lie group G lies in the fact that one can study directly bifurcations of orbits of critical points. We do not work with the quotient space Ω/G. The choice of equivariant bifurcation theory seems to be the best adapted to our theory. In this article we apply only the $SO(3)$-invariant potentials, because the Newton potential is $SO(3)$-invariant when we consider motion of bodies in $\mathbb{R}^3$. But the abstract results presented in this section are more general, i.e. we consider symmetries of an arbitrary compact Lie group G. Finally, it is worth pointing out that the assumptions of theorems presented in this section are easy to verify. Notions of invariant potentials and families of invariant potentials have been formulated in Definitions A.1 and A.2, respectively. A notion of a local and a global bifurcation of orbits of critical points of invariant potentials has been formulated in Definitions A.3 and A.4, respectively. These definitions of topological bifurcations presented in this article are in the spirit of the notion of a bifurcation considered in [14–18]. The necessary condition for the existence of a local bifurcation of orbits of critical points has been formulated in Lemma A.1. Sufficient conditions for the existence of a local and a global bifurcation of orbits of critical points have been formulated in Theorems A.1 and A.2 respectively. Additionally, we obtain conditions for localization of a local and a global bifurcation orbits in Corollaries A.1 and A.2, respectively.

Finally, we underline that one can prove much stronger equivariant bifurcation theorems than those presented in the Appendix. But theorems formulated in this section are strong enough to study central configurations of the N-body problem.

2. Preliminaries and statement of the main results

In this section we study bifurcations of classes of central configurations of the N-body problem from known families of classes of spatial central configurations, whose existence has been proved in [19]. We underline that in our approach the masses of bodies $m = (m_1, \dots, m_N) \in (0, +\infty)^N$ are treated as parameters.

Let us consider $\mathbb{W} = \mathbb{R}^3$ as a natural orthogonal representation of the group $SO(3)$. By $\mathbb{V}$ we denote a $3N$-dimensional representation of $SO(3)$ which is a direct sum of N-copies of the representation $\mathbb{W}$: i.e.

$$\mathbb{V} = \underbrace{\mathbb{W} \oplus \cdots \oplus \mathbb{W}}_{N\text{-times}}.$$

The action of the group $SO(3)$ on $\mathbb{V}$ is given by $SO(3) \times \mathbb{V} \ni (g, q) = (g, (q_1, \dots, q_N)) \to g \cdot q = (g \cdot q_1, \dots, g \cdot q_N) \in \mathbb{V}$. Denote by $\mathbb{R}$ a one-dimensional trivial $SO(3)$-representation. Then the $SO(3)$-action on the representation $\mathbb{V} \oplus \mathbb{R}$ is defined by $SO(3) \times (\mathbb{V} \oplus \mathbb{R}) \ni (g, (q, \omega)) \Longrightarrow (g \cdot q, \omega) \in \mathbb{V} \oplus \mathbb{R}$. Define $\Omega = \{q \in \mathbb{V} : q_i \neq q_j, \text{ for } i \neq j\}$ and note that Ω is an open and $SO(3)$-invariant subset of the representation $\mathbb{V}$. Let $U : \Omega \times (0, +\infty)^N \to \mathbb{R}$ be the Newtonian potential, i.e. $U(q, m) = \sum_{1 \leq i < j \leq N} \frac{m_i m_j}{\|q_i - q_j\|}$, where we assume that the gravitational constant $G = 1$. It is easy to verify that $U \in C^{\infty}_{SO(3)}(\Omega \times (0, +\infty)^N, \mathbb{R})$, i.e. U is a smooth family of $SO(3)$-invariant potentials, see Definition A.1.

The equations of motion of the N-body problem are given by

$$\ddot{q}(t) = \nabla_q U(q(t), m). \tag{1}$$

Later in this paper we will be working with two important concepts: a central configuration and the size of the system formed for the N-particles at any time t, measured by the moment of inertia.