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Multiple bifurcation analysis in a NDDE arising from van der Pol's equation with extended delay feedback[☆]

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ABSTRACT

Van der Pol's equation with extended delay feedback control is considered, which is equivalent to a system of neutral differential–difference equations (NDDEs). Fold bifurcation and Hopf bifurcation in this NDDE are studied by the formal adjoint theory, the center manifold theorem and the normal form method. These methods are also first employed in studying the Bogdanov–Takens singularity of NDDE. Bifurcation sets theoretically indicate the existence of a homoclinic orbit and the coexistence of three periodic solutions, which are all illustrated by the numerical methods. The coexistence of three stable periodic solutions and the existence of stable torus near the Hopf–fold and Hopf–Hopf bifurcations are also illustrated, respectively.

1. Introduction

Van der Pol's equation is widely studied by many authors since it was first formulated for an electrical circuit with a triode valve; see for example [1–6]. This equation is perhaps best known as a prototype system exhibiting limit cycle oscillations. Thus, stabilizing these periodic solutions (or control, we say) is an important issue in the nonlinear dynamics, especially, in the control theory. Meanwhile, exploring more complicated dynamical behavior is also an interesting problem in the anti-control theory; see [7]. Delay feedback control is an effective approach to stabilize the periodic solutions. The general linear delay feedback scheme includes the so-called direct delay feedback and the delay feedback of difference-type, i.e., $kx(t-\tau)$ and $k[x(t-\tau)-x(t)]$, where x is the state variable of van der Pol's equation, k is the strength of the feedback and τ is the time delay.

Van der Pol's equation with delay feedback is studied a lot. Atay [1] studied van der Pol's equation with feedback $kx(t-\tau)$, and obtained that both stable and unstable periodic solutions may appear in this system. Wei and Jiang [6,8] studied the Hopf bifurcation and the existence of codimension two bifurcation in van der Pol's equation with direct delay feedback. Ma et al. [5] studied the equation with a difference-type feedback, where the feedback strength depends on the time lag. They mainly studied the Hopf–Hopf bifurcation and observed some interesting phenomena such as stable tori and chaos, etc. In [9], Pyragas studied an effective feedback control, namely, the extended delay feedback control, which is a modification of the traditional delay feedback control technique that allows one to stabilize unstable periodic orbits over a large domain of parameters. This method is established as the method of extended time delay autosynchronization, which is actually a continuous feedback loop incorporating information from a sequence of previous states, with the corresponding flow chart shown in Fig. 1.

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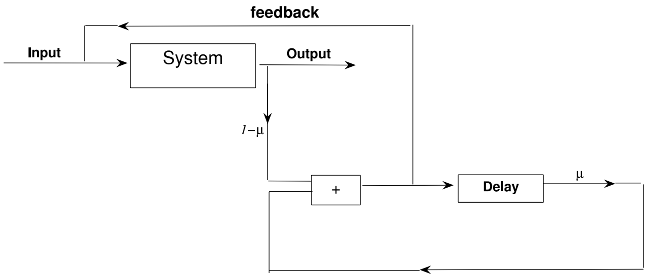

Fig. 1. The flow chart of extended delay feedback control.

Van der Pol's equation with extended delay feedback is of the form:

$$\ddot{x} + \varepsilon(x^2 - 1)\dot{x} + x = \varepsilon k\vartheta(t) + g(x) \tag{1}$$

where $\varepsilon > 0$. The feedback term includes two parts: k is the strength of the feedback $\vartheta(t)$, which is the linear part in the feedback signal. $\vartheta(t)$ depends on the current state and a sequence of past states, which is defined by

$$\vartheta(t) = (1-\mu)x(t) + \mu\vartheta(t-\tau), \tag{2}$$

with $0 < \mu < 1$. This is actually $\vartheta(t) = (1-\mu)x(t) + \mu(1-\mu)x(t-\tau) + \mu^2(1-\mu)x(t-2\tau) + \cdots$. Thus Eq. (1) can be viewed as a functional differential equation with infinite time delay. The other nonlinear term $g(x)$ is at least C^3 smooth. For convenience, we denote

$$g(0) = g'(0) = 0, \qquad g''(0) = 2\kappa_1, \qquad g'''(0) = 6\kappa_2. \tag{3}$$

Actually, Eq. (1) is equivalent to

$$\mu\ddot{x}(t-\tau) + \mu\varepsilon(x^2(t-\tau) - 1)\dot{x}(t-\tau) + \mu x(t-\tau) = \mu\varepsilon k\vartheta(t-\tau) + \mu g(x(t-\tau)). \tag{4}$$

Using (1)–(4), we have

$$\begin{aligned} &\ddot{x} - \mu\ddot{x}(t-\tau) + \varepsilon(x^2-1)\dot{x} - \mu\varepsilon(x^2(t-\tau)-1)\dot{x}(t-\tau) + x - \mu x(t-\tau) \\ &\quad = \varepsilon k(1-\mu)x(t) + g(x) - \mu g(x(t-\tau)). \end{aligned} \tag{5}$$

This is a neutral differential–difference equation (NDDE) of second order.

Introduce a new variable $y(t) = \dot{x}(t)$, then (5) becomes a system of NDDEs

$$\begin{cases} \dot{x} = y \\ \dot{y} - \mu\dot{y}(t-\tau) = [-1 + \varepsilon k(1-\mu)]x + \varepsilon y + \mu x(t-\tau) - \varepsilon\mu y(t-\tau) \\ \qquad -\varepsilon x^2 y + \varepsilon\mu x^2(t-\tau)y(t-\tau) + g(x) - \mu g(x(t-\tau)). \end{cases} \tag{6}$$

Usually, investigating an equation with infinite time delay requires a phase space defined by Hale and Kato [10], which is quite difficult. Moreover, the bifurcation theory for the equation with infinite delay has not been well studied yet because of the complexity of its phase space. Motivated by such a consideration we will study the equivalent NDDE to obtain the dynamics of system (1).

This paper mainly aims to give bifurcation analysis in NDDE (6). As mentioned above, van der Pol's equation with delay feedback was studied a lot, some of which were from the viewpoint of bifurcation analysis. We find certain systems with extended delay feedback can often be transformed into NDDEs. Thus bifurcation analysis in a NDDE, of codimension one and codimension two, is necessary to reveal the dynamical behavior in van der Pol's equation with extended delay feedback. In this paper we will focus on the fold, Hopf and Bogdanov–Takens bifurcations in Eq. (6).

As is known to all, the most famous NDDE model is the lossless transmission line. We refer the readers to [11–16] and the references cited therein. Theoretically analyzing a NDDE is more difficult than a retarded differential–difference equation (RDDE), because the solution map of NDDE is not compact but an α-contraction (see [11]). Generally, the local Hopf bifurcation analysis of a NDDE is motivated by [13,17–22]. Here Wei and Ruan [13] gives a local stability theorem by analyzing the associated characteristic equations. Wang and Wei [23,24] calculate the normal form of a NDDE model near the Hopf bifurcation point based on the formal adjoint theory given in [17,19,20] and the algorithm given in [18,21,22], respectively.

There are two main parts in this paper. One is the analysis of fold and Hopf bifurcations in NDDE (6). The other is to study the codimension two bifurcations. In recent years, some results about codimension two bifurcations in RDDE have been published. Jiang et al. [25–27] studied the codimension two bifurcations in van der Pol's equation with nonlinear delay feedback, which includes Bogdanov–Takens bifurcation, Hopf–transcritical bifurcation, and Hopf–pitchfork bifurcation, respectively. Zhang et al. [28] studied the Bogdanov–Takens bifurcation in a delayed predator–prey diffusion system with a functional response. So far, general research about codimension two bifurcations focuses on the RDDE. Compared with a