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Remarks on semilinear parabolic systems with terms concentrating in the boundary

Marcone C. Pereira *

Escola de Artes, Ciências e Humanidades, Universidade de São Paulo, São Paulo, Brazil

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ABSTRACT

We are concerned with the asymptotic behavior of a dynamical system generated by a family of semilinear parabolic systems with reaction and potential terms concentrating in a neighborhood of a portion of the boundary. Assuming that this neighborhood shrinks to this section as a parameter ϵ goes to zero, we exhibit the limit problem and show continuity of the flux as well as upper and lower semicontinuity of the family of global attractors with respect to ϵ using an appropriated functional setting on suitable conditions for the system. It is worth noting that oscillatory behavior to the neighborhood as ϵ goes to zero is also allowed providing a large range of applications.

1. Introduction

In this paper we are interested in discussing the behavior of the asymptotic dynamics generated by a dissipative semilinear parabolic system with nonlinear boundary conditions when some reaction and potential terms are concentrated in a narrow neighborhood of a portion Γ_0 of the boundary that shrinks to it when a small positive parameter ϵ goes to zero.

Our nonlinear reaction–diffusion model is supposed to represent several interactions among agents, which can be cells, amount of chemical, density of animals and so on, whose some one of reactions occur only in an extremely thin neighborhood near this region border Γ_0. It is worth noting that our model enables this narrow strip also bring forward some kind of oscillating behavior modeling complex regions of interactions.

Potential applications of the results presented here include management and control of aquatic ecological systems where one finds localized concentrations in connection with boundary complexity. For instance, we may mention [1–5] where theoretical and practical aspects of mathematical modeling applied to limnology, oceanography and transitional water systems have been deeply investigated.

Of course, we are dealing with a singular partial differential equation brought out by thickness and oscillating behavior of the ϵ-neighborhood where the reactions take place. We will see that in some sense, this singular boundary value problem can be approximated by a semilinear nonlinear reaction–diffusion system, also with nonlinear boundary condition, where the geometric oscillating behavior of the reaction neighborhood is exhibited as a flux condition and a potential term on the portion Γ_0 of the domain border. The limit problem obtained is not singular being an option to replace the original one when the ϵ parameter is close to zero, that is, when the small strip is very narrow and rough. Also, it displays some features of the original system pointing out their emergent properties giving us conditions to get the qualitative behavior of the modeled problem.

* Tel.: +55 11 2935 7169.
E-mail address: marcone@usp.br.

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We observe that this sort of modeling was initially introduced in [6], where a linear elliptic problem was considered. There, they restrict themselves to consider a narrow neighborhood without oscillatory behavior. Later, the asymptotic behavior of the dynamics of a nonlinear parabolic equation was analyzed in [7,8], where the upper semicontinuity of attractors was proved. We recall that an attractor is a compact invariant set which attracts the flow for all bounded sets of the phase space. In fact, it contains all the asymptotic dynamics of the system and all global bounded solutions lie in the attractor.

Analogous arguments from [6] has been employed in [9,10], where a reaction–diffusion problem with delay has been studied. In those works, the boundary of the domain is suppose to be smooth. Results from [6] also have been adapted in [11] to a nonlinear elliptic problem posed on a Lipschitz domain presenting a highly oscillatory behavior on a neighborhood of the boundary using some ideas from [12,13], where elliptic and parabolic equations defined in thin domains with a highly oscillatory behavior have been extensively investigated. We have also shown in [14] the upper and lower semicontinuity of the dynamics of the scalar case of this model, that is, when (2.1) is just one equation and just one agent is considered to react.

Our goal here is to extend the results from [14] to a semilinear parabolic system, with nonlinear boundary condition. In fact, we are interested in discussing here a more effective model with reaction and potential terms concentrating on boundary. We consider the situation in which several agents are interacting near an oscillating narrow strip very close to some portion of the border. Our approach will be somewhat different from the one in [7,8] and closer to the one in [14], where some abstract results on continuity of attractors from [15] were properly applied.

2. Notation and main results

To be more precise, let us introduce the semilinear parabolic model. We begin setting the open bounded a C^2-regular domain $\Omega \subset \mathbb{R}^2$ with boundary $\partial\Omega$ satisfying $\partial\Omega = \Gamma_0 \cup \Gamma_1$ where $\{\Gamma_0, \Gamma_1\}$ is a regular partition of the boundary, that is, $\Gamma_0 \cap \Gamma_1 = \emptyset$. Note that either Γ_0 or Γ_1 could be empty. Now, to set the narrow neighborhood we will use two nonnegative parameters, namely, ϵ and α. ϵ will represent the thickness of the strip, and α its oscillatory behavior. We take a function $G_\epsilon(\cdot) = G_{\epsilon,\alpha}(\cdot)$ satisfying $0 < G_0 \leqslant G_\epsilon(\cdot) \leqslant G_1$ for positive constants G_0 and G_1, which is high oscillating as the small positive parameter $\epsilon \to 0$ and $\alpha > 0$. We establish this by expression

$$G_\epsilon(x) = G(x, x/\epsilon^\alpha), \quad \epsilon > 0, \text{ and } \alpha \geq 0,$$

where the function $G : (0, 1) \times \mathbb{R} \mapsto \mathbb{R}$ is a positive smooth function such that $y \to G(x, y)$ is $l(x)$-periodic in y for each x, with period $l(x)$ uniformly bounded in $(0, 1)$, that is, $0 < l_0 < l(\cdot) < l_1$. Next, let $x, y \in C^2([0, 1])$ such that the curve $\zeta(s) = (x(s), y(s))$, $s \in [0, 1]$, is a C^2-parametrization of the portion $\Gamma_0 \subset \partial\Omega$ with $\|\zeta'(s)\| = 1$, for all $s \in [0, 1]$. We also assume that $N(\zeta(s)) = (y'(s), -x'(s))$ is the unit outward normal vector to Γ_0. Finally, we can define our narrow oscillating strip by

$$\omega_\epsilon = \omega_{\epsilon,\alpha} = \left\{\xi \in \mathbb{R}^2 : \xi = \zeta(s) - tN(\zeta(s)),\ s \in [0, 1] \text{ and } 0 \leqslant t < \epsilon\, G_\epsilon(s)\right\},$$

for $\epsilon > 0$ sufficiently small, say $0 < \epsilon \leqslant \epsilon_0$, and $\alpha \geq 0$.

This way of setting thickness and oscillation, let us denote the functions and the strip only with respect to ϵ. Indeed, we enable oscillations and thickness just assigning $\alpha > 0$ and taking ϵ very close to zero. It is clear that the subset ω_ϵ defines a neighborhood of Γ_0 in $\bar{\Omega}$ that collapses to the portion Γ_0 when the parameter ϵ goes to zero since the function G_ϵ is uniformly bounded for all $\alpha \geq 0$. Also, we observe that the "inner boundary" of ω_ϵ, given by

$$\left\{\xi \in \mathbb{R}^2 : \xi = \zeta(s) - \epsilon\, G_\epsilon(s)\, N(\zeta(s)),\ s \in [0, 1]\right\},$$

presents a high oscillatory behavior as $\alpha \gg 0$ and ϵ is close to zero. Note that our assumptions allow the case without oscillating, achieved when $\alpha = 0$, and defined by $G_\epsilon(x) = G(x, x)$ independent of ϵ, as well as, the purely periodic behavior given, for example, by $G_\epsilon(x) = 2 + \sin(x/\epsilon^\alpha)$ with $\alpha > 0$. Observe that we also include the case where G_ϵ sets up a neighborhood wherein the oscillation period, the amplitude and the strip profile is modulated with respect to the variable $x \in (0, 1)$ such as $G_\epsilon(x) = K(x) + A(x)\sin(P(x)/\epsilon^\alpha)$ with $\alpha > 0$ and $K, A, P : [0, 1] \mapsto \mathbb{R}^+$. See Figs. 1 and 2 that illustrate the graph of an accepted function G_ϵ, and the oscillating strip $\omega_\epsilon \subset \Omega$ for the purely periodic case with $\alpha > 0$.

In this work, we are interested in the long time behavior of the solutions of the following weakly coupled nonlinear reaction–diffusion system of the form

$$\begin{cases} u^\epsilon_t - \operatorname{div}(a(\cdot)\nabla u^\epsilon) + \lambda u^\epsilon + \dfrac{1}{\epsilon}\mathcal{X}_\epsilon\, V^\epsilon(\cdot)\, u^\epsilon = f(\cdot, u^\epsilon) + \dfrac{1}{\epsilon}\mathcal{X}_\epsilon\, g(\cdot, u^\epsilon) & \text{in } \Omega \\ \dfrac{\partial u^\epsilon}{\partial n_a} = h(\cdot, u^\epsilon) \quad \text{on } \Gamma_0 \\ u^\epsilon = 0 \quad \text{on } \Gamma_1 \end{cases} \tag{2.1}$$

where $u = (u_1, \ldots, u_m)^\intercal$, for some $m \in \mathbb{N}$, $a(x) = \operatorname{diag}(a_1(x), \ldots, a_m(x))$, $a_i \in \mathcal{C}^1(\bar{\Omega})$, $a_i(x) > m_0 > 0$ for $x \in \Omega$, $\operatorname{div}(a(\cdot)\nabla u^\epsilon) = (\operatorname{div}(a_1(\cdot)\nabla u^\epsilon_1), \ldots, \operatorname{div}(a_m(\cdot)\nabla u^\epsilon_m))^\intercal$, $\frac{\partial u}{\partial n_a} = (\frac{\partial u_1}{\partial n_{a_1}}, \ldots, \frac{\partial u_m}{\partial n_{a_m}})^\intercal$ with $\frac{\partial u_i}{\partial n_{a_i}} = \langle a_i \nabla u_i, n\rangle$ for $1 \leq i \leq m$, n denotes the unit outward normal vector to $\partial\Omega$, $\mathcal{X}_\epsilon \in L^\infty(\Omega)$ is the characteristic function of the set ω_ϵ and λ is a suitable real number. The nonlinearities $f = (f_1, \ldots, f_m)^\intercal$, $g = (g_1, \ldots, g_m)^\intercal$, and $h = (h_1, \ldots, h_m)^\intercal : \mathcal{O} \times \mathbb{R}^m \mapsto \mathbb{R}^m$ are smooth functions where $\mathcal{O} \subset \mathbb{R}^2$ is an open set containing $\bar{\Omega}$.