Nonlinear Analysis: Real World Applications 14 (2013) 1235–1246

Contents lists available at SciVerse ScienceDirect

Nonlinear Analysis: Real World Applications

journal homepage: www.elsevier.com/locate/nonrwa

Effects of cross-diffusion and heterogeneous environment on positive steady states of a prey–predator system

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ARTICLE INFO

Article history:
Received 26 December 2011
Accepted 24 September 2012

Keywords:
Cross-diffusion
Heterogeneous environment
Bifurcation
Stationary solution

ABSTRACT

In this paper, we consider the positive steady state problem of a spatially heterogeneous cross-diffusion prey–predator model with modified Leslie–Gower and Holling-Type II schemes. The heterogeneity here is created by a protection zone for the prey. By the bifurcation method and a priori estimates, we discuss the existence and non-existence of positive steady states. Moreover, uniqueness and stability of positive steady states for small birth rate of the predator are shown as well as the asymptotic behavior of positive steady states when some coefficients tend to infinity. Our result reveals that large cross-diffusion in a *heterogeneous environment* has a profound effect on the positive steady state set, and the bifurcation continuum of positive steady states changes from a bounded one to an unbounded one as the cross-diffusion varies from 0 to a large number for suitable ranges of the parameter. Whereas it has little effect in the *homogeneous environment*. The impact of the protection zone is also quite important, and it deduces a critical number for the birth rate λ of the prey which determines the bifurcation continuum of positive steady states to be bounded or unbounded. Furthermore, the modified Leslie–Gower term yields essentially different results of positive solutions from the Leslie–Gower term.

1. Introduction

In this paper, we shall consider the following prey–predator model with cross-diffusion and protection for the prey:

$$\begin{cases} u_t = \Delta\left[(1+\tau\rho(x)v)\,u\right] + u\left(\lambda - u - \dfrac{m(x)v}{k+u}\right), & x\in\Omega,\ t>0,\\ v_t = \Delta v + v\left(\mu - \dfrac{cv}{d+u}\right), & x\in\Omega\setminus\bar{\Omega}_0,\ t>0,\\ \partial_\nu u = 0,\quad x\in\partial\Omega,\ t>0, & \partial_\nu v = 0,\quad x\in\partial\Omega\cup\partial\Omega_0,\ t>0,\\ u(x,0)=u_0(x)\ge 0,\quad x\in\Omega, & v(x,0)=v_0(x)\ge 0,\quad x\in\Omega\setminus\bar{\Omega}_0. \end{cases} \tag{1.1}$$

Here Ω is a bounded domain in $\mathbb{R}^n (n \ge 1)$, Ω_0 is a smooth domain satisfying $\bar{\Omega}_0 \subset \Omega$; ν is the outward unit normal vector on the boundary and $\partial_\nu = \partial/\partial\nu$; $u(x, t)$ and $v(x, t)$ represent the population density of the prey and predator, respectively; $\rho(x) = 1$ and $m(x) = m > 0$ in $\bar{\Omega} \setminus \Omega_0$, whereas $\rho(x) = m(x) = 0$ in Ω_0; Ω_0 is a protection for the prey u, the predator v cannot enter Ω_0, and so $\partial_\nu v = 0$ on $\partial\Omega_0$, while the prey can enter and leave Ω_0 freely; $\tau \ge 0$ is the cross-diffusion coefficient and represents the sensitivity of the prey species to the population pressure from the predator species; $\lambda > 0$

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1468-1218/$ – see front matter
doi:10.1016/j.nonrwa.2012.09.015

and $\mu > 0$ denote the birth rate of the respective species; $k > 0$ (resp. $d > 0$) measures the extent to which the environment provides protection to u (resp. v); $m > 0$ in $\bar{\Omega} \setminus \Omega_0$ is the maximum value which per capita reduction rate of u can attain; $c > 0$ has a similar meaning to m. The system is self-contained, and there is no flux on the boundary $\partial\Omega$.

There is a nonlinear diffusion term $\tau\Delta[\rho(x)vu]$ in the first equation of (1.1), which is usually referred to as the cross-diffusion term and reflects the fact that the movement of prey u is affected by the population pressure from the predator v. This was first proposed by Shigesada et al. [1] to model the habitat segregation phenomena between two competing species, one can see [2] for more backgrounds for cross-diffusion. Many researchers have payed attention to studying the effect of cross-diffusion from various aspects since the pioneering work [1], including the global existence of solutions and positive steady state problem, one can refer to [3,8,10,9,4,5,7,11,14,12,13,6,15–17] and references therein for more details.

The reaction–diffusion system with spatially homogeneous coefficients have been widely and extensively studied since the 1970s, in particular, the ODE and diffusive versions of (1.1) have been studied in [18] and [19]. While some interesting papers investigating the heterogeneous effect of environment have appeared in recent years. Dancer and Du [20] and Du et al. [21,23–27,22] have studied the effects of the heterogeneous environment caused by the protection zone or the degeneracy of some intra-specific pressures. The effects of spatially heterogeneous birth rates have been shown by Dockery et al. [28] and Hutson et al. [29,31,30,32] for some diffusive competition models. We note that cross-diffusion is not included in the above work. Taking cross-diffusion into account, there is little work. Oeda[33] studied a cross-diffusive prey–predator model with a protection zone, and obtained some interesting results. One can also refer to [34] and [35] for a discussion of the heterogeneous environment which is not caused by the above cases.

The main purpose of this paper is to consider the positive steady state problem of (1.1). That is to say, we shall study the following problem

$$\begin{cases} \Delta[(1+\tau\rho(x)v)u] + u\left(\lambda - u - \dfrac{m(x)v}{k+u}\right) = 0, & x \in \Omega, \\ \Delta v + v\left(\mu - \dfrac{v}{1+u}\right) = 0, & x \in \Omega_1 = \Omega \setminus \bar{\Omega}_0, \\ \partial_\nu u = 0, \quad x \in \partial\Omega, & \partial_\nu v = 0, \quad x \in \partial\Omega_1, \end{cases} \tag{1.2}$$

where we take $c = d = 1$ without loss of generality. From the biological viewpoint, we tend to obtain the bifurcation structure of the positive solution set of (1.2), where a positive solution corresponds to a coexistence steady state of the prey and predator.

One of our main goals in the paper is to investigate the effects of cross-diffusion and a heterogeneous environment on the positive solution set of (1.2). By Remark 3.7 in Section 3, we can see that cross-diffusion in the homogeneous environment ($\Omega_0 = \emptyset$) has little effect on (1.2). That is, when $\Omega_0 = \emptyset$, the bifurcation curve of positive solutions of (1.2) remains bounded as the cross-diffusion varies from 0 to a large value. While if the environment is heterogeneous ($\Omega_0 \neq \emptyset$), cross-diffusion has a profound impact on the structure of the positive solution set. In the heterogeneous case, there exists a critical value $\lambda^*(\tau, \Omega_0)$ such that a bounded continuum $\mathscr{C}$ bifurcates from the semitrivial solution $(\lambda, 0)$ at $\mu = 0$ and joins with $(0, \mu)$ at some point $\mu = \mu_1$ if $0 < \lambda < \lambda^*(\tau, \Omega_0)$; while if $\lambda \geq \lambda^*(\tau, \Omega_0)$, the bounded continuum $\mathscr{C}$ bifurcating from $(\lambda, 0)$ at $\mu = 0$ becomes unbounded and tends to ∞ as the bifurcation parameter μ tends to ∞. Furthermore, Lemma 2.1 deduces that $\lambda^*(\tau, \Omega_0) \to 0$ as $\tau \to \infty$ and $\lambda^*(0, \Omega_0) = \lambda_1^D(\Omega_0)$, where $\lambda_1^D(\Omega_0)$ denotes the principal eigenvalue of $-\Delta$ over Ω_0 subject to Dirichlet boundary condition. Then as τ is sufficiently large such that λ satisfies $\lambda^*(\tau, \Omega_0) \leq \lambda < \lambda_1^D(\Omega_0)$, one sees that large cross-diffusion causes the original bounded continuum $\mathscr{C}$ to be an unbounded one. The bifurcation structure of the positive solution set totally changed as τ changes from 0 to a large number. The effect of a heterogeneous environment is the same as that discussed in [25] and [33]. Precisely, there exists a critical patch size of the protection zone for every model, and the critical size is determined by an equation of the form $\lambda = \lambda^*(\tau, \Omega_0)$, which corresponds to the value $\lambda_1^D(\Omega_0)$ in [25] as $\tau = 0$. If the protection zone is below its critical size, namely $0 < \lambda < \lambda^*(\tau, \Omega_0)$, (1.2) has no positive solution for large $\mu > 0$; while if the protection zone is above the critical size, namely $\lambda \geq \lambda^*(\tau, \Omega_0)$, then there exits coexistence state of the prey and predator even though the predator v has a large birth rate μ.

The other goal of this paper is to study the asymptotic behavior of positive solutions of (1.2). Our result shows that if $0 < \lambda \leq \lambda_1^D(\Omega_0)$, then all positive solutions (u_τ, v_τ) of (1.2) tend to a uniform steady state, that is, $(0, \mu)$; while if $\lambda > \lambda_1^D(\Omega_0)$, the prey and predator species become spatially segregated. The asymptotic behavior of positive solution u as $\mu \to \infty$ is similar to that in case $\tau \to \infty$, while $v \to \infty$ uniformly in $\bar{\Omega}_1$ as $\mu \to \infty$. Furthermore, Theorem 4.3 shows that the positive non-constant solution of (1.2) is unique and linearly stable for small $\mu > 0$.

We next compare the results when $\tau = 0$ with those of [24], where the same kind of protection zone is introduced. In [24], the authors considered the following system

$$\begin{cases} -\Delta u = u(\lambda - u - b(x)v), & x \in \Omega, \\ -\Delta v = v\left(\mu - \dfrac{v}{u}\right), & x \in \Omega_1, \\ \partial_\nu u = 0, \quad x \in \partial\Omega, & \partial_\nu v = 0, \quad x \in \partial\Omega_1, \end{cases} \tag{1.3}$$