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Validity of the Cahn–Hilliard approximation for modulations of slightly unstable pattern in the real Ginzburg–Landau equation

Wolf-Patrick Düll *

Institut für Analysis, Dynamik und Modellierung, Universität Stuttgart, 70569 Stuttgart, Germany

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ABSTRACT

In order to describe slow modulations in time and space of slightly unstable spatially periodic stationary solutions of pattern forming reaction–diffusion systems, the Cahn–Hilliard equation can be derived via multiple scaling analysis as a formal approximation equation. By proving estimates between the approximations obtained via this procedure and the exact solutions of the original system the validity of the Cahn–Hilliard equation as an approximation equation can be rigorously justified. This has been done for a class of one-dimensional reaction–diffusion systems in Düll (2007) [18]. In this paper, we provide a simpler, more elementary and shorter validity proof for the case of the real Ginzburg–Landau equation as an original pattern forming system by exploiting the S^1-symmetry of the real Ginzburg–Landau equation.

1. Introduction and result

Many pattern forming systems – for example, classical hydrodynamic stability problems like the Rayleigh–Bénard convection or the Taylor–Couette problem and reaction–diffusion systems like the Brusselator – show the following universal behavior. Due to energy dissipation there exists a trivial ground state, e.g. a trivial steady-state solution, which is stable as long as a certain system parameter is below a critical value. However, if this system parameter goes beyond this critical value, the ground state loses its stability and more complicated solutions occur, which bifurcate from the trivial solution. To describe the evolution of the amplitude of slow modulations in time and space of the underlying bifurcating pattern the so-called modulation equations can be derived by multiple scaling analysis.

The prototype of such a modulation equation is the real Ginzburg–Landau equation

$$\partial_T U = \partial_X^2 U + U - U|U|^2 \tag{1}$$

with $X \in \mathbb{R}$, $T \geq 0$ and $U(X, T) \in \mathbb{C}$. A mathematical theory of the reduction of pattern forming systems to the Ginzburg–Landau equation has been developed by several authors; cf. [1–10].

The real Ginzburg–Landau equation possesses a family of periodic stationary solutions

$$U = U_{\text{per}}[q, \phi_0](X) = \sqrt{1 - q^2} e^{iqX + i\phi_0} \tag{2}$$

with $\phi_0 \in \mathbb{R}$ and $|q| \leq 1$. These periodic stationary solutions also have a physical meaning. For example, in the context of the Rayleigh–Bénard convection problem they yield a leading-order description of the temperature fluctuations (around

* Tel.: +49 71168565377; fax: +49 71168565535.
E-mail address: duell@mathematik.uni-stuttgart.de.

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the temperature field of the purely conducting ground state) which are generated by convection rolls bifurcating from the ground state; cf. [11, Ch. 1.1,1.3,3.1, 4, 9] and the references therein.

The spatially periodic stationary patterns modeled by U_{per} are stable with respect to perturbations for $q^2 < 1/3$ and unstable for $q^2 > 1/3$. To describe formally slow modulations in time and space of U_{per} near the boundaries of the so-called Eckhaus-stable region, i.e., $q^2 \approx 1/3$, cf. [12], the Cahn–Hilliard equation can be derived via multiple scaling analysis. Thus, the Cahn–Hilliard equation describes a secondary bifurcation; cf. [11,13].

In the following we review the derivation of the Cahn–Hilliard equation. In order to do so we introduce the polar coordinates

$$U(X,T) = r(X,T)e^{i\phi(X,T)}$$

in the real Ginzburg–Landau equation and obtain

$$\partial_T r = \partial_X^2 r + r - (\partial_X \phi)^2 r - r^3, \tag{3}$$

$$\partial_T \phi = \partial_X^2 \phi + \frac{2(\partial_X r)(\partial_X \phi)}{r}. \tag{4}$$

To analyze the real Ginzburg–Landau equation it is often convenient to use polar coordinates; cf. [14].

Since on the right-hand side of (3)–(4) only derivatives of the phase ϕ occur we can replace the equation for the phase by an equation for the local wavenumber $\tilde{\psi} = \partial_X \phi$ and get

$$\partial_T r \ = \partial_X^2 r + r - (\tilde{\psi})^2 r - r^3 \tag{5}$$

$$\partial_T \tilde{\psi} \ = \partial_X^2 \tilde{\psi} + 2\partial_X \frac{(\partial_X r)\tilde{\psi}}{r}. \tag{6}$$

In these coordinates, the spatially periodic patterns U_{per} discussed above turn into the constant functions ($r = \sqrt{1-q^2}$, $\tilde{\psi} = q$). To study the evolution of perturbations of U_{per} we fix q and introduce the deviations ($s := r - \sqrt{1-q^2}$, $\psi := \tilde{\psi} - q$) from the chosen pattern ($r = \sqrt{1-q^2}$, $\tilde{\psi} = q$). We obtain

$$\partial_T s = \partial_X^2 s - 2(1-q^2)s - 2q\sqrt{1-q^2}\psi - \sqrt{1-q^2}\psi^2 - 2q\psi s - 3\sqrt{1-q^2}s^2 - \psi^2 s - s^3, \tag{7}$$

$$\partial_T \psi = \partial_X^2 \psi + 2\partial_X \frac{(\partial_X s)(q+\psi)}{\sqrt{1-q^2}+s}. \tag{8}$$

Linearizing at $s = \psi = 0$ yields the system

$$\partial_T s = \partial_X^2 s - 2(1-q^2)s - 2q\sqrt{1-q^2}\psi \ , \tag{9}$$

$$\partial_T \psi = \partial_X^2 \psi + 2q(1-q^2)^{-1/2}\partial_X^2 s. \tag{10}$$

This system has plane wave solutions of the form

$$(s, \psi) = (s_k, \psi_k)e^{ikX + \hat{\mu}_j(k)T}$$

with $(s_k, \psi_k) \in \mathbb{C}^2$ and

$$\hat{\mu}_{1,2}(k) = -(1-q^2) - k^2 \pm \sqrt{(1-q^2)^2 + 4q^2k^2}.$$

We always have that $\hat{\mu}_2(k)$ is strictly negative for $k \in \mathbb{R}$, but $\hat{\mu}_1(0) = 0$. Since (9)–(10) is symmetric with respect to the mapping $X \mapsto -X$ the functions $\hat{\mu}_{1,2}$ are even. Therefore, we have $\partial_k \hat{\mu}_1(0) = 0$, which implies that the stability of U_{per} is determined by the sign of $\partial_k^2 \hat{\mu}_1(0)$. We have $\partial_k^2 \hat{\mu}_1(0) < 0$ for $q^2 < 1/3$, that means that U_{per} is spectrally stable, and $\partial_k^2 \hat{\mu}_1(0) > 0$ for $q^2 > 1/3$, that means that U_{per} is (sideband or Eckhaus) unstable; cf. [12,3] for more details.

In order to describe the nonlinear dynamics of slow modulations in time and space near stable or slightly unstable stationary solutions U_{per}, we introduce a small perturbation parameter $0 < \delta \ll 1$, insert the ansatz

$$\psi(X,T) = \delta^\alpha A(\delta X, \delta^\beta T) = \delta^\alpha A(\xi, \tau).$$
$$s(X,T) = \delta^\alpha B(\delta X, \delta^\beta T) = \delta^\alpha B(\xi, \tau)$$

into (7)–(8) and equate the terms with the lowest power of δ. This leads to the following formal approximation equations. If $|q| < \sqrt{1/3}$, $\alpha = 1$ and $\beta = 2$ one obtains the heat equation

$$\partial_\tau A = \frac{1-3q^2}{1-q^2}\partial_\xi^2 A, \tag{11}$$