Nonlinear Analysis: Real World Applications 15 (2014) 38–41

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

journal homepage: www.elsevier.com/locate/nonrwa

Monotone solutions to a nonlinear integral equation of convolution type

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ARTICLE INFO

Article history:
Received 23 April 2013
Accepted 8 May 2013

ABSTRACT

We provide sufficient conditions under which the unique solution to a nonlinear integral equation of convolution type is monotonic. As a consequence, we obtain information about the asymptotic behaviour of the solution in situations where previous results fail to apply.

1. Introduction

In this paper we investigate the asymptotic behaviour of the solutions to the integral equation

$$\Phi(u(t)) = L(t) + \int_0^t P(t-s)u(s)\,\mathrm{d}s, \quad t \geq 0, \tag{1}$$

where $L(t)$ and $P(t)$ are continuous positive functions on the interval $[0, \infty)$ and $u(t)$ is the unknown positive function. In the linear case $\Phi(u) \equiv u$ this equation arises in population theory, in the theory of industrial displacements and the general theory of self-renewing aggregates [1,2]. In this setting an explicit formula for the solution is attainable by means of the Laplace transform, and the asymptotic behaviour of this solution can be investigated (see the discussion in [3]). The setting $\Phi(u) = u^2$ models water percolation through a porous material placed on a solid, non-permeable base, with the solution $u(t)$ corresponding to the height of the water above the base at time t, cf. [4,5]. Despite the lack of known explicit solutions in nonlinear settings, criteria ensuring that $\lim_{t\to\infty} u(t)$ exists and is finite were obtained in [6] for $\Phi(u) = u^2$, in [3] for $\Phi(u) = u^p$ with $p > 1$, and in [7] for a more general class of nonlinearities. The standard assumptions on Φ to ensure the existence and uniqueness of a continuous solution $u : [0, \infty) \to [0, \infty)$ are:

(I) The continuous function $\Phi : [0, \infty) \to [0, \infty)$ is locally expansive on $(0, \infty)$, that is, for every compact $K \subset (0, \infty)$ there exists a positive constant c_K such that

$$c_K \leq \frac{|\Phi(u_1) - \Phi(u_2)|}{|u_1 - u_2|} \quad \text{for all } u_1,\ u_2 \in K \text{ with } u_1 \neq u_2;$$

(II) $\lim_{u\to\infty} \frac{\Phi(u)}{u} = \infty$;

(III) $\lim_{u\to 0} \frac{\Phi(u)}{u} = 0$.

(see [7]). Note that these assumptions ensure that Φ is strictly increasing, with the inverse $\Phi^{-1} : [0, \infty) \to [0, \infty)$.

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http://dx.doi.org/10.1016/j.nonrwa.2013.05.003

Eq. (1) has interesting asymptotic properties, as shown in [6,7,3]. From [7], we know that (1) has a unique nonnegative solution provided Φ satisfies the assumptions (I)–(III). Now define

$$I := \int_0^\infty P(s)\,\mathrm{d}s; \qquad \lim_{t\to\infty} L(t) = L.$$

It turns out that the solution to (1) is bounded if and only if I is finite and the function L is bounded (see [7]). Moreover, if $\lim_{t\to\infty} L(t) = L$ exists and if the equation

$$\Phi(x) - Ix - L = 0$$

has a unique positive solution $A_{L,I}$, then, as shown in [7],

$$\lim_{t\to\infty} u(t) = A_{L,I}.$$

In this note we investigate the asymptotic behaviour of the unique solution of (1) using a different approach in order to cover cases where the above condition of unique solvability of $\Phi(x) - Ix - L = 0$ in $(0, \infty)$ does not necessarily hold. More precisely, we study the existence of monotone solutions. In the final section we illustrate by means of examples the applicability of our results to situations where previous results are inconclusive.

2. Monotone solutions

Let $u(0) = U_0$, so that (1) evaluated at $t = 0$ yields

$$U_0 = \Phi^{-1}(L(0)). \tag{2}$$

We are now in a position to state and prove our main results.

Theorem 2.1. *Suppose that Φ is continuously differentiable and satisfies* (I)–(III). *Furthermore, assume that L, P are continuously differentiable and P is non-increasing. If*

$$L'(t) + U_0 P(t) < 0$$

for $t \in [0, \infty)$, then $u(t)$ is strictly decreasing on $[0, \infty)$.

Proof. Our hypotheses ensure that the function $u(t)$ is differentiable. Differentiating (1) with respect to t yields

$$\Phi'(u(t))u'(t) = L'(t) + P(0)u(t) + \int_0^t P'(t-s)u(s)\,\mathrm{d}s. \tag{3}$$

When we evaluate (3) at $t = 0$ we have

$$\Phi'(u(0))u'(0) = L'(0) + P(0)u(0).$$

It is then clear that

$$\Phi'(u(0))u'(0) < 0$$

by our assumption. Note that we also have that $\Phi'(u) > 0$ for $u > 0$. Therefore

$$u'(0) < 0. \tag{4}$$

From (4) we deduce that there exists a number $\epsilon > 0$ such that $u'(t) < 0$ for all $t \in [0, \epsilon)$.

Assume that there exists a number $T > 0$ such that $u'(t) < 0$ for all $t \in [0, T)$ and with $u'(T) = 0$. We evaluate (3) at $t = T$ to obtain

$$0 = L'(T) + P(0)u(T) + \int_0^T P'(T-s)u(s)\,\mathrm{d}s.$$

Since $u(t)$ is strictly decreasing on $[0, T)$, we deduce

$$0 \le L'(T) + P(0)u(T) + u(T)\int_0^T P'(T-s)\,\mathrm{d}s. \tag{5}$$

A change of variable in the above integral yields

$$0 \le L'(T) + P(0)u(T) + u(T)\int_0^T P'(y)\,\mathrm{d}y.$$

We then observe that

$$\begin{aligned} 0 &\le L'(T) + P(0)u(T) + u(T)P(T) - u(T)P(0) = L'(T) + u(T)P(T) \\ &\le L'(T) + u(0)P(T) = L'(T) + U_0 P(T). \end{aligned}$$