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Multiplicity and uniqueness of positive solutions for elliptic equations with nonlinear boundary conditions arising in a theory of thermal explosion

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ABSTRACT

In this paper we study a model of thermal explosion which is described by positive solutions to the boundary value problem

$$\begin{cases} -\Delta u = \lambda f(u), & x \in \Omega, \\ \mathbf{n} \cdot \nabla u + c(u)u = 0, & x \in \partial\Omega, \end{cases}$$

where $f, c : [0, \infty) \to (0, \infty)$ are C^1 and $C^{1,\gamma}$ non decreasing functions satisfying $\lim_{u\to\infty} \frac{f(u)}{u} = 0$, Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and $\lambda > 0$ is a parameter. Using the method of sub and super-solutions we show that the solution of this problem is unique for large and small values of parameter λ, whereas for intermediate values of λ solutions are multiple provided nonlinearity f satisfies some natural assumptions. An example of such nonlinearity which is most relevant to applications and satisfies all our hypotheses is $f(u) = \exp[\frac{\alpha u}{\alpha+u}]$ for $\alpha \gg 1$.

1. Introduction

The problem of thermal explosion, spontaneous initiation of a rapid combustion process, is a classical problem of combustion theory which has been studied for over 80 years. The modeling of this process traces back to pioneering works of Semenov, Frank-Kamenetskii and Zeldovich [1–3]. In a rather general setting the problem of spontaneous ignition can be formulated as a following initial value problem:

$$\begin{cases} T_t - \Delta T = \lambda f(T), & (t, x) \in (0, \infty) \times \Omega, \\ \mathbf{n} \cdot \nabla T + c(T)T = 0, & (t, x) \in (0, \infty) \times \partial\Omega, \\ T(0, x) = 0, & x \in \partial\Omega. \end{cases} \tag{1.1}$$

Here, T is an appropriately normalized temperature distribution in a bounded smooth domain $\Omega \subset \mathbb{R}^N, N \geq 1$, which evolves in time due to thermal diffusivity and chemical reaction. The latter is described by chemical kinetics term $f(T)$. The most common example of chemical reaction term is Arrhenius law in which case

$$f(T) = \exp\left(\frac{\alpha T}{\alpha + T}\right), \tag{1.2}$$

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where the parameter $\alpha > 0$ (usually large) is a scaled activation energy. More generally and as we assume throughout of this paper the function f satisfies following two hypotheses:

(H1) $f : [0, \infty) \to (0, \infty)$ is a C^1 non decreasing function,

and

(H2) $\lim_{s\to\infty} \frac{f(s)}{s} = 0$.

The parameter $\lambda > 0$ is a scaling parameter and can be associated with the size of the domain Ω which grows (in a physical space) as λ increases. To account substantial interaction with the surroundings, heat-loss conditions are imposed on a boundary $\partial\Omega$ with outward normal $\mathbf{n}$, where the heat loss parameter $c(T)$ satisfies:

(H3) $c : [0, \infty) \to (0, \infty)$ is a $C^{1,\gamma}$ non decreasing function with $\gamma \in (0, 1)$.

Physically this assumption means that a heat loss through the boundary is always present and increases for higher temperatures. Finally, initial normalized temperature is assumed to be equal to the one of the surroundings which is set to be equal to zero.

It is well known that a long time behavior of solutions for the problem (1.1) is fully determined by its stationary solutions, that is solutions of the following problem

$$\begin{cases} -\Delta u = \lambda f(u), & x \in \Omega, \\ \mathbf{n} \cdot \nabla u + c(u)u = 0, & x \in \partial\Omega. \end{cases} \tag{1.3}$$

Indeed, a direct application of the parabolic comparison principle [4] shows that solutions of the problem (1.1) approach a *minimal solution* of the problem (1.3) as $t \to \infty$ provided the latter exists (see Remark 2.1 in next section for more details). Here and below a minimal solution $u^\sharp$ of the problem (1.3) is a function which verifies (1.3) (possibly in a weak sense) and satisfies $u^\sharp \le u$ in $\bar{\Omega}$ for any function u which solves (1.3). One can show that such a solution exists and is well defined as long as (1.3) has a solution [5]. Thus, comparison principle implies that a limiting temperature distribution is the minimal possible stationary temperature distribution. As a result, the analysis of thermal explosion described by the problem (1.1) reduces to the analysis of stationary temperature distributions described by the problem (1.3).

Let us point out that the nonlinear heat loss boundary condition in (1.1) and (1.3) which is considered in this paper is not very typical for classical combustion problems but is relevant to some more recent applications. Indeed, in a classical theory of thermal explosion when say modeling ignition in a combustion chamber heat loss on the boundary is so strong and intrinsic time scales are so small that one may assume that the temperature on the boundary of the domain is equal to the one of the surroundings which lead to a Dirichlet boundary conditions $T = 0$ on $\partial\Omega$, so called cold boundary condition. This condition can be obtained from the one in (1.1) and (1.3) by formally setting $c = \infty$. This case, therefore, corresponds to an infinite heat loss on the boundary. The case of cold boundary conditions was studied quite extensively in the literature (see [6–16]). In some real world applications, however, simplifications of classical approach discussed above cannot be adopted. Such applications include problems of safe storage of energetic materials and nuclear waste [17] or even raw garbage [18]. For these applications induction time (time prior to ignition) can vary from several hours to months, whereas in the classical theory of thermal explosion induction time is typically a fraction of a second. It is clear that over such a long time period the boundary of reactive material will be preheated to a temperature significantly higher than the one of the surroundings. As a result, conventional cold boundary conditions are not applicable anymore and a heat loss boundary condition should be used instead to properly describe the thermal equilibrium of the boundary.

As we mentioned earlier, solutions of the problem (1.3) with Dirichlet boundary conditions (problem (1.3) with $c = \infty$) were studied by several authors. The behavior of solutions for this problem, which always exist, critically depends on properties of the nonlinearity and the value of scaling parameter λ. In the case when the nonlinear term $f(u)$ resembles the one corresponding to Arrhenius kinetics (given by Eq. (1.2)) with high activation energy ($\alpha \gg 1$), generically (1.3) has multiple solutions for a certain range of the parameter λ. Specifically, it was shown in [8,9,11,12,19,20,16] that for small and large values of λ the solution of (1.3) with $c = \infty$ is unique, whereas for intermediate values of λ there are at least three solutions. The typical diagram of L^∞ norm of such solutions is given on Fig. 1. This picture has a very natural physical interpretation. Recall that λ is a scaling parameter which scales the size of the domain. For small domains with the size smaller than the one corresponding to λ_* the heat loss on the boundary suppressed the chemical reaction inside the domain and thus temperature stays at relatively small values. We note that in this regime even substantial energy deposition into the system does not lead to explosion but rather to a slow burning and eventual quenching. For domains of intermediate size ($\lambda_* \le \lambda \le \lambda^*$) thermal explosion is still impossible, however substantial deposition of the energy into the system may lead to ignition associated with rapid transition from minimal to maximal solution. For large domains corresponding to $\lambda > \lambda^*$ the situation is opposite. The chemical reaction inside the domain dominates the boundary heat loss which leads to spontaneous explosion. In other words, thermal explosion occurs in a region of λ's where minimal solutions obtained as continuation of minimal solution corresponding to $\lambda = 0$ cease to exist. To reflect this fact it is convenient to introduce a notion of $*$-minimal solution. We say that $u^\sharp_{\lambda'}$ belongs to a family of *$*$-minimal solutions* if $u^\sharp_{\lambda'}$ is a minimal solution of (1.3) and minimal solutions $u^\sharp_\lambda$ are continuous with respect to λ on $(0, \lambda')$. Using this term one can say that thermal explosion occurs exclusively due to the absence of $*$-minimal solution for the problem (1.3) for $\lambda > \lambda^*$.