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On an induction–conduction PDEs system in the harmonic regime

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ABSTRACT

We study the existence of weak solutions to a nonlinear coupled parabolic–elliptic system arising in the heating industrial process of a steel workpiece. The unknowns are the electric potential, the magnetic vector potential and the temperature. The different time scales related to the electric potential and the magnetic vector potential versus the temperature lead us to introduce the harmonic regime. This yields to a new system of nonlinear partial differential equations.

1. Introduction

This work is devoted to analyze the existence of weak solutions to a nonlinear coupled system of partial differential equations which constitutes the mathematical modeling of the heating industrial process by induction–conduction of a steel workpiece. The main goal of heat treating of steel is to attain a satisfactory hardness on certain critical parts of the workpiece while keeping the rest ductile. In this paper, the workpiece represents an automobile steering rack. The rack is a solid cylinder with a tooth line profile as shown in Fig. 1.

Among the different hardening surface procedures, we are interested in an induction–conduction industrial procedure. In this way, a copper inductor is put in contact with the rack as it is shown in Fig. 2. Then a high frequency alternating current is switched on and flows through the coil made up by the workpiece itself and the copper inductor. An alternating magnetic field is generated which in its turn induces eddy currents bringing about heat (Joule's effect) just where it is needed. Once the desired high level of temperature is reached at certain critical parts along the rack, the supplied electric current is switched off, and the workpiece is quenched in order to cool it down rapidly.

The supplied density current flow is modeled through a Neumann boundary condition on a fictitious cross-section Γ cutting across the copper inductor (see Fig. 2).

The heating–cooling industrial processes are governed by a coupled nonlinear system of partial differential equations and ordinary differential equations. The mathematical description of the setting corresponding to Fig. 2 can be found in [1] together with some numerical simulations. As it is shown in this figure, the inductor and the workpiece share the common boundary S. The heating model (1)–(8) reflects this fact mainly in the expression of the Joule term, which takes the form $\sigma(\theta)|\boldsymbol{\mathcal{A}}_{,t} + \nabla\phi|^2$. Note that in the case of direct current, $\boldsymbol{\mathcal{A}}$ does not depend on t, and the model reduces to the so-called thermistor problem [2–4]; on the other hand, if the inductor does not touch the workpiece, then the electric potential does not depend on x on the workpiece and the Joule term becomes $\sigma(\theta)|\boldsymbol{\mathcal{A}}_{,t}|^2$. This situation is considered, for instance, in [5,6] where the model is also rewritten in the harmonic regime (see Section 3). The novelty of this paper is the mathematical

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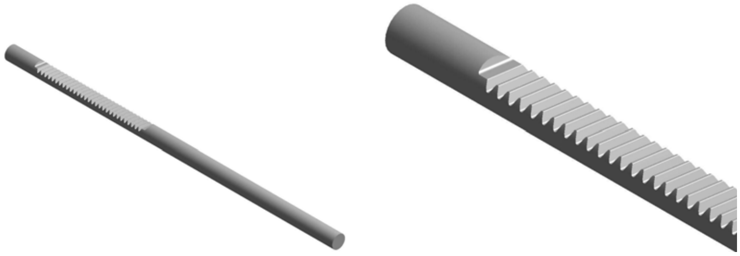

Fig. 1. Automobile steering rack.

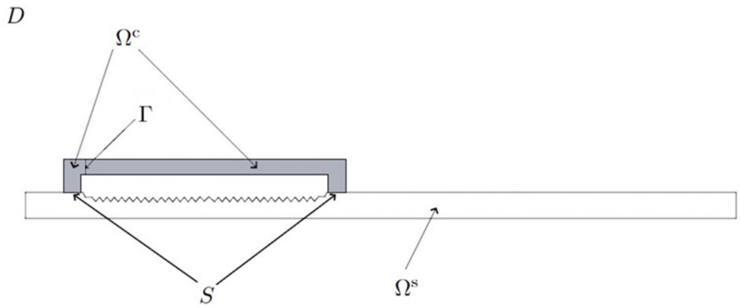

Fig. 2. Open sets D, Ω^s and Ω^c, contact surfaces S, and the auxiliary cross-section Γ.

analysis and resolution of the induction–conduction heating process corresponding with the situation described in Fig. 2 and the unknowns are expressed in the harmonic regime. The resulting model is given in (10)–(17). The analysis of this system is more difficult than the one of the thermistor problem since the Joule term cannot be written as the divergence of an L^2 function [2]. The same comment can be said about the induction case [6] as the Joule heating yields a more regular term.

This work is organized as follows. In Section 2 we establish the mathematical description for the heating process. Section 3 is devoted to introduce the harmonic regime. In Section 4 we set up the notation used in this paper; this leads to the introduction of some functional spaces. We also enumerate the hypotheses on data, recall certain compactness results, give the notion of weak solution adapted to our problem and state the main result. Finally, Section 5 develops the proof of the existence result; it is split into three steps, namely setting of the approximate problems, derivation of estimates, and passing to the limit and conclusion.

2. Setting of the problem

Our main task is to analyze the existence of weak solutions of a simplified model which does not take into account mechanical effects.

To this end, let Ω, $D \subset \mathbb{R}^3$ be bounded, connected and Lipschitz-continuous open sets such that $\bar{\Omega} \subset D$, $\Omega = \Omega^c \cup \Omega^s \cup S$ is the set of conductors, Ω^c is the copper inductor, Ω^s is the steel workpiece containing a toothed part to be hardened, Ω^c and Ω^s being open sets, and $S = \bar{\Omega}^c \cap \bar{\Omega}^s$ is the surface contact between Ω^c and Ω^s, $\Omega^c \cap \Omega^s = \varnothing$ (see Fig. 2).

In our setting the high frequency current density supplied through the workpiece is about 80 kHz. Then we may neglect the electric displacement term in the set of Maxwell's equations. A high frequency current is supplied during a time interval $[0, T]$ passing through the set of conductors $\Omega = \Omega^s \cup \Omega^c \cup S$. Due to its shape, the set of conductors constitutes a coil which in its turn induces electromagnetic eddy currents inside the workpiece. The combined effect of both conduction and induction through the workpiece results in an energy dissipation (Joule's heating) leading to an increase in temperature in the critical parts of the workpiece to be hardened. This heating process takes about $T = 5.5$ s. Once the desired temperature is reached, the current is switched off and the cooling stage begins by spraying the workpiece with water. This process is called aquaquenching. Let $\phi: \Omega \times [0, T] \mapsto \mathbb{R}$ be the electric potential, $\boldsymbol{\mathcal{A}}: D \times [0, T] \mapsto \mathbb{R}^3$ the magnetic vector potential, and $\theta: \Omega \times [0, T] \mapsto \mathbb{R}$ the temperature. Neglecting mechanical effects, the heating process is described by the following system of elliptic–parabolic PDEs [1,7–13]:

$$\nabla \cdot [\sigma(\theta)\nabla\phi] = 0 \quad \text{in } \Omega_T = \Omega \times (0, T), \tag{1}$$

$$\frac{\partial \phi}{\partial n} = 0 \quad \text{on } \partial\Omega \times (0, T), \tag{2}$$

$$\left[\sigma(\theta)\frac{\partial \phi}{\partial \nu}\right]_\Gamma = j_S \quad \text{on } \Gamma \times (0, T), \tag{3}$$