Nonlinear Analysis: Real World Applications 15 (2014) 80–88

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

journal homepage: www.elsevier.com/locate/nonrwa

Pullback attractor for a non-linear evolution equation in elasticity[☆]

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ARTICLE INFO

Article history:
Received 29 March 2013
Accepted 10 June 2013

ABSTRACT

We prove the existence of a pullback attractor for a non-autonomous fourth order evolution equation arising in the field of phase transitions and elasticity theory. The existence of several families of bounded absorbing sets is first proved in several spaces, and owing to the compactness of some inclusions between Sobolev spaces, we can then ensure the existence of a family of compact absorbing sets in the pullback sense and, as a consequence, the existence of a pullback attractor.

1. Introduction

In [1–3] the asymptotic behavior of the following evolution equation has been studied:

$$\begin{cases} u_t = -\varepsilon^2 u_{xxxx} + \dfrac{1}{2} W''(u_x) u_{xx}, & \\ u = u_{xx} = 0, & \text{on } \partial I, \\ u(0,t) = u_0, & \end{cases} \tag{1.1}$$

where $u : I \to \mathbb{R}, 0 < \varepsilon \ll 1, I = (0, 1)$ and $W(p) = (p^2 - 1)^2$ is the so called double well potential. The Eq. (1.1) represents the L^2-gradient dynamics of the non-convex energy:

$$F_\varepsilon(u) = \frac{1}{2}\varepsilon^2 \int_I u_{xx}^2 dx + \frac{1}{2}\int_I W(u_x) dx. \tag{1.2}$$

The equation arises in the field of elasticity theory, phase transition and image processing.

The functional (1.2) with $\varepsilon = 0$ is a simple model describing microstructures that arise from solid–solid phase transitions in certain elastic crystals (see [4,5]) such as In–Th Cu–Al–Ni Ni–Ti. These materials present a variety of microstructures which are important for technological application in the context of material theory (shape memory effects, pseudoelasticity, etc.). From a physical point of view, the functional (1.2) (with $\varepsilon = 0$) describes the elastic energy required for deformation of the crystal, while the function $W(\cdot)$ is the stored energy density functions describing the properties of the material. The addition of the regularizing term, depending on the second derivative of u and on the small parameter ε, solves the problem of non-uniqueness of minimizers (and of the ill-posedness of the equation of the associated L^2 gradient dynamics) without

☆ Partly supported by FEDER and Ministerio de Economía y Competitividad (Spain) under grant MTM2011-22411.

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http://dx.doi.org/10.1016/j.nonrwa.2013.06.002

changing the relevant macroscopic features of the model. It is natural to consider the associated dynamical problem (see [6,7]), that is, the study of the gradient flow of the energy (1.2) which can be written in the form of system (1.1).

In [8] they studied the global dynamics of (1.1). By numerical experiments they show the existence of three well separated time scales with peculiar dynamical behavior. In a first time scale of order $t > T_\varepsilon = O(\varepsilon^2)$ the energy of the initial data is drastically reduced and microstructures (see [4]) appear in the regions where u_x falls in the non-convex region of W. In a second time scale of order $t > T = O(1)$ the equation exhibits a heath equation-like behavior in the regions without microstructures, while the dynamic is slow in the regions with microstructures. In the third time scale $t > \frac{1}{T_\varepsilon} = O(\varepsilon^{-2})$ the equation shows a finite dimensional character, i.e. the solutions are approximately the union of consecutive segments. In particular the existence of an inertial manifold has been conjectured in [8]. In the papers [1,2] the finite dimensional reduction of the system has been proved. In particular the authors prove the existence, giving an estimate of the dimension, of the global attractor, exponential attractor and inertial manifold.

Non-linear dynamical systems are subjected to aleatoric influences, that is the reason why in the present paper we consider a small non-autonomous perturbation of Eq. (1.1) and consider the effects on the dynamics and in particular on the third time scale.

The theory of global attractors has been generalized in the case of non-autonomous systems by introducing the concept of uniform attractor (see, for instance Chepyzhov and Vishik [9]). However this concept of uniform attractor lacks of the property of invariance which may be a serious inconvenience in the analysis of the long time behavior of the system. For this reason, an alternate possibility has been developed within the framework of pullback attractors theory (see Kloeden and Rasmussen [10] and Carvalho et al. [11] and some other papers included in their bibliography sections). One advantage of the latter is that it is not necessary to impose very restrictive assumptions on the time dependent terms in the equation. Another advantage of this theory comes from the fact that, originally, the concept of pullback attractor was introduced in the field of random dynamical systems (see Schmalfuss [12] and Crauel and Flandoli [13]), and for this reason it is a suitable concept to treat random and non-autonomous features in the models.

The study of non-autonomous or stochastic perturbation for a fourth order evolution equation is quite recent. An important fourth order PDE is the Cahn–Hilliard equation that describes phase transitions in a binary metal alloy. In [14] the author studied the existence of an exponential attractor for the viscous Cahn–Hilliard equation:

$$\frac{\partial}{\partial t}[v + \varepsilon(-\Delta v)] + \Delta^2 v - \Delta W(v) = m(t),$$

where the non-autonomous perturbation satisfies

$$\int_I m(t)dx = 0.$$

Stochastic perturbation has also been studied. For instance, in [15] it is considered a stochastic perturbation of the type:

$$v_t + \Delta(\Delta v - W(u)) = \sigma(u)\dot{B}_t,$$

proving the existence and uniqueness of solutions. B_t represents a Wiener process and σ is a bounded Lipschitz function. In the article [16] they showed the existence of a random attractor for a stochastic Cahn–Hilliard equation with dynamics and stochastic boundary conditions, while the stochastic three-dimensional Lagrangian averaged Navier–Stokes equations are analyzed in [17].

These arguments suggest the importance of considering perturbation of Eq. (1.1), in particular there is a deep relation between Cahn–Hilliard equation and Eq. (1.1), in fact if v is the solution of the Cahn–Hilliard equation:

$$v_t + \Delta[\varepsilon^2 \Delta v - W'(v)] = 0,$$

with Neumann boundary condition

$$\frac{\partial}{\partial n}v = \frac{\partial}{\partial n}\Delta v = 0, \quad x \in \{0, 1\},$$

then

$$u(x) = \int_0^x v(s)ds,$$

is the solution of Eq. (1.1) with boundary conditions

$$u = u_{xx} = 0, \quad x \in \{0, 1\}.$$

The rest of the paper is organized as follows: in Section 2 we present some useful preliminary estimates and remarks, while Section 3 contains the main results, that is, the existence of several absorbing sets and the existence of a pullback attractor in $L^2(I)$.