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Predictor homotopy analysis method: Two points second order boundary value problems

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ABSTRACT

The purpose of the present paper is to give a proof for a systematic method namely, predictor homotopy analysis method (PHAM) to predict the multiplicity of the solutions of two points nonlinear second order boundary value problems of type $u'' = f\left(x,\ u, u'\right)$. We present briefly the method first, and then some theorems are given to clarify this issue that, how one can predict the existence of the multiple solutions and obtain them simultaneously as well.

1. Introduction

Predictor homotopy analysis method (PHAM) has been proposed recently to predict the multiplicity of the solutions of some nonlinear boundary value problems [1–3]. This method has in fact a new point of view to the well-known homotopy analysis method (HAM) [4–47] so that it uses *convergence controller parameter* in erudite way by adding so-called *prescribed parameter* to the problem. This technique can be easily applied on nonlinear ordinary differential equations with boundary conditions. This method, besides predicting the multiplicity of the solutions of the nonlinear differential equations, calculates effectively the all branches of the solutions (provided that, there exist such solutions for the problem) analytically at the same time. In this manner, for practical use in science and engineering, this method might give a new unfamiliar class of solutions which is of fundamental interest and furthermore, the proposed approach convinces to apply it on nonlinear equations by today's powerful symbolic software programs so that it does not need tedious stages of evaluation and can be used without studying the whole theory.

2. Description of the method

To illustrate the procedure consider the following nonlinear differential equation:

$$u'' = f\left(x,\ u, u'\right), \qquad \alpha < x < \beta \tag{1}$$

with boundary conditions

$$a_1 u\left(\alpha\right) + a_2 u'(\alpha) = a, \qquad b_1 u\left(\beta\right) + b_2 u'(\beta) = b \tag{2}$$

where f is general nonlinear operator. The crucial step of the technique is that the boundary value problem (1)–(2) should be replaced by the equivalent initial value problem so that the conditions (2) involve an unknown parameter δ (*prescribed parameter*) as follows

$$u\left(\alpha\right) = g_1\left(\delta\right), \qquad u'\left(\alpha\right) = g_2\left(\delta\right), \quad \text{and} \quad b_1 u\left(\beta\right) + b_2 u'(\beta) = b \tag{3}$$

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where $b_1u(\beta)+b_2u'(\beta) = b$ is the forcing condition that comes from the original conditions (2), and $g_1(\delta)$ and $g_2(\delta)$ satisfy $a_1g_1(\delta) + a_2g_2(\delta) = a$. Now, homotopy analysis method is applied on the problem (1) with the conditions (3) except the forcing condition i.e.

$$u'' = f\left(x, u, u'\right), \qquad \alpha < x < \beta \tag{4}$$

$$u(\alpha) = g_1(\delta), \qquad u'(\alpha) = g_2(\delta). \tag{5}$$

2.1. Zero-order deformation equation

We suppose that all the solutions $u = u(x)$ of the problem (4) can be expressed by the set of base functions $\{\omega_i(x), i = 0, 1, 2, \ldots\}$ in the form

$$u = u(x) = \sum_{n=0}^{+\infty} a_n\omega_n(x) \tag{6}$$

where a_n are the coefficients to be determined.

Let $u_0(x, \delta)$ denote an initial approximation guess of the exact solution $u(x)$ which satisfies the initial conditions (5) automatically. Also, as that is well-known in the frame of HAM, assume that $\hbar \neq 0$ denotes the convergence-controller parameter, $H(x) \neq 0$ an auxiliary function, and $\mathcal{L}$ an auxiliary linear operator of type second order. Now using $p \in [0, 1]$ as an embedding parameter, we construct the general zero-order deformation equation and the corresponding boundary conditions as follows:

$$(1-p)\,\mathcal{L}\left[\varphi(x, \delta; p) - u_0(x, \delta)\right] = p\hbar H(x)\mathcal{N}\left[\varphi(x, \delta; p)\right] \tag{7}$$

$$u(\alpha) = g_1(\delta), \qquad u'(\alpha) = g_2(\delta) \tag{8}$$

where $\varphi(x, \delta; p)$ is an unknown function to be determined and

$$\mathcal{N}\left[\varphi(x, \delta; p)\right] = \frac{\partial^2\varphi(x, \delta; p)}{\partial x^2} - f\left(x, \varphi(x, \delta; p), \frac{\partial\varphi(x, \delta; p)}{\partial x}\right).$$

When $p = 0$, the zero order deformation equation (7) becomes

$$\mathcal{L}\left[\varphi(x, \delta; 0) - u_0(x, \delta)\right] = 0 \tag{9}$$

which gives $\varphi(x, \delta; 0) = u_0(x, \delta)$. When $p = 1$, the Eq. (7) leads to

$$\mathcal{N}\left[\varphi(x, \delta; 1)\right] = 0 \tag{10}$$

which is exactly the same as the original equation (1) provided that $\varphi(x, \delta; 1) = u(x, \delta)$.

We now expand the function $\varphi(x, \delta; p)$ in a Taylor series to the embedding parameter p. This Taylor expansion can be written in the form

$$\varphi(x, \delta; p) = u_0(x, \delta) + \sum_{m=1}^{m=+\infty} u_m(x, \delta)p^m \tag{11}$$

where

$$u_m(x, \delta) = \frac{1}{m!}\frac{\partial^m\varphi(x, \delta; p)}{\partial p^m}, \qquad m = 0, 1, 2, \ldots. \tag{12}$$

As it is well known in the frame of HAM, when the linear operator $\mathcal{L}$, the initial approximation $u_0(r, \delta)$, the auxiliary parameter $\hbar \neq 0$, and the auxiliary function $H(x) \neq 0$ are chosen properly, the series (11) converges for $p = 1$, and thus

$$u(x, \delta) = u_0(x, \delta) + \sum_{m=1}^{m=+\infty} u_m(x, \delta) = \sum_{n=0}^{+\infty} a_n\omega_n(x) \tag{13}$$

will be the solution of the nonlinear problem (4) and (5) as will be proved later.

2.2. High-order deformation equation

Assume that the linear operator $\mathcal{L}$, the initial approximation $u_0(x, \delta)$, and the auxiliary function $H(x) \neq 0$ are chosen properly (it is worth mentioning here that, $\hbar \neq 0$, so-called convergence-controller parameter will be determined later), the