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Existence of weak solutions for a diffuse interface model of non-Newtonian two-phase flows

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ABSTRACT

We consider a phase field model for the flow of two partly miscible incompressible, viscous fluids of non-Newtonian (power law) type. In the model it is assumed that the densities of the fluids are equal. We prove the existence of weak solutions for general initial data and arbitrarily large times with the aid of a parabolic Lipschitz truncation method, which preserves solenoidal velocity fields and was recently developed by Breit, Diening, and Schwarzacher.

1. Introduction

We consider the flow of two macroscopically immiscible, incompressible non-Newtonian fluids. In contrast to classical sharp interface models, a partial mixing of the fluids is taken into account, which leads to a so-called diffuse interface model. This has the advantage that flows beyond the occurrence of topological singularities e.g. due to droplet collision or pinch-off can be described. More precisely we consider

$$\rho\partial_t \mathbf{v} + \rho\mathbf{v}\cdot\nabla\mathbf{v} - \operatorname{div}\mathbf{S}(c,\mathbf{D}\mathbf{v}) + \nabla p = -\kappa \operatorname{div}(\nabla c\otimes\nabla c), \tag{1.1}$$

$$\operatorname{div}\mathbf{v} = 0, \tag{1.2}$$

$$\partial_t c + \mathbf{v}\cdot\nabla c = m\Delta\mu, \tag{1.3}$$

$$\mu = \kappa^{-1}\phi(c) - \kappa\Delta c \tag{1.4}$$

in $Q_T = \Omega\times(0,T)$, where $\Omega\subseteq\mathbb{R}^d$, $d\geq 2$, is a bounded domain and $T\in(0,\infty)$. Here $\mathbf{v}$ is the mean velocity, $\mathbf{D}\mathbf{v} = \frac{1}{2}(\nabla\mathbf{v}+\nabla\mathbf{v}^T)$, p is the pressure, c is an order parameter related to the concentration of the fluids e.g. the concentration difference or the concentration of one component, and ρ is the density of the fluids, which is assumed to be constant. Moreover, $\mathbf{S}(c,\mathbf{D}\mathbf{v})$ is the viscous part of the stress tensor of the mixture to be specified below, $\kappa > 0$ is a (small) parameter, which is related to the "thickness" of the interfacial region, $\Phi\colon\mathbb{R}\to\mathbb{R}$ is a homogeneous free energy density and $\phi = \Phi'$ and μ is the chemical potential. Capillary forces due to surface tension are modeled by an extra contribution $\kappa\nabla c\otimes\nabla c := \kappa\nabla c(\nabla c)^T$ in the stress tensor leading to the term on the right-hand side of (1.1). Moreover, we note that in the modeling diffusion of the fluid components is taken into account. Therefore $m\Delta\mu$ is appearing in (1.3), where $m>0$ is a constant mobility coefficient.

We close the system by adding the boundary and initial conditions

$$\mathbf{v}|_{\partial\Omega} = 0 \quad \text{on } \partial\Omega\times(0,T), \tag{1.5}$$

$$\mathbf{n}\cdot\nabla c|_{\partial\Omega} = \mathbf{n}\cdot\nabla\mu|_{\partial\Omega} = 0 \quad \text{on } \partial\Omega\times(0,\infty), \tag{1.6}$$

$$(\mathbf{v},c)|_{t=0} = (\mathbf{v}_0,c_0) \quad \text{in } \Omega. \tag{1.7}$$

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Here $\mathbf{n}$ denotes the exterior normal at $\partial\Omega$. We note that (1.1) can be replaced by

$$\rho\partial_t \mathbf{v} + \rho\mathbf{v}\cdot\nabla\mathbf{v} - \operatorname{div}\mathbf{S}(c, \mathbf{Dv}) + \nabla g = \mu\nabla c \tag{1.8}$$

with $g = p + \frac{\kappa}{2}|\nabla c|^2 + \kappa^{-1}\Phi(c)$ since

$$\mu\nabla c = \nabla\left(\frac{\kappa}{2}|\nabla c|^2 + \kappa^{-1}\Phi(c)\right) - \kappa\operatorname{div}(\nabla c\otimes\nabla c). \tag{1.9}$$

In the case of Newtonian fluids, i.e., $\mathbf{S}(c, \mathbf{Dv}) = \nu(c)\mathbf{Dv}$ for some positive viscosity coefficient $\nu(c)$, the model was first discussed by Hohenberg and Halperin [1]. Later it was derived in the framework of rational continuum mechanics by Gurtin, Polignone, Viñals [2]. The latter derivation can be easily modified to include a suitable non-Newtonian behavior of the fluids. If e.g. $\mathbf{S}(c, \mathbf{Dv})$ is chosen such that $\mathbf{S}(c, \mathbf{Dv}) : \mathbf{Dv} \geq 0$, the local dissipation inequality, which yields thermodynamical consistency, remains valid. For results on existence of weak and strong solutions in the case of Newtonian fluids we refer to Starovoitov [3], Boyer [4], and A. [5]. First analytic results for the system (1.1)–(1.4) for non-Newtonian fluids of power-law type were obtained by Kim, Consiglieri, and Rodrigues [6]. The authors proved existence of weak solutions if $q \geq \frac{3d+2}{d+2}$, $d = 2, 3$, where q is the power describing the growth of the stress tensor with respect to $\mathbf{Dv}$. For this range of q monotone operator techniques can be applied. Moreover, in the case $d = 3$ and $2 \leq q < \frac{11}{5}$ the authors prove existence of measure-valued solutions. Grasselli and Pražak [7] discussed the longtime behavior of solutions of (1.1)–(1.4) in the case $q \geq \frac{3d+2}{d+2}$, $d = 2, 3$ assuming periodic boundary conditions and a regular free energy density. For the same range of q results on existence of weak solutions with a singular free energy density f and the longtime behavior were obtained by Bosia [8] in the case of a bounded domain in $\mathbb{R}^3$.

The goal of this article is to extend the existence result to lower values of q in order to include the physically important case of shear thinning flows. In the case of a single fluid existence of weak solutions for power-law type fluids was proved for the case $q > \frac{2d}{d+2}$, $d \geq 2$, by D., Růžička, and Wolf [9]. The proof is based on a parabolic Lipschitz truncation method and a careful decomposition of the pressure, which is needed since the Lipschitz truncation used does not preserve the divergence freeness of a velocity field. Recently a parabolic Lipschitz truncation method, which keeps divergence free velocity fields divergence free, was developed by Breit, D. and Schwarzacher [10]. In the present article we will use this method in order to prove existence of weak solutions to (1.1)–(1.7) if $\mathbf{S}(c, \mathbf{Dv})$ is of power law type with an exponent $q > \frac{2d}{d+2}$. Precise assumptions are made in the following.

For simplicity we assume that $\kappa = \rho = 1$, but all results are true for general (fixed) $\kappa, \rho > 0$. Moreover, we assume:

Assumption 1.1. Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded domain with C^3-boundary and let $\Phi \in C([a, b]) \cap C^2((a, b))$ be such that $\phi = \Phi'$ satisfies

$$\lim_{s\to a}\phi(s) = -\infty, \qquad \lim_{s\to b}\phi(s) = \infty, \qquad \phi'(s) \geq -\alpha$$

for some $\alpha \in \mathbb{R}$. Let $m > 0$ and let $\mathbf{S}\colon [a, b]\times\mathbb{R}^{d\times d} \to \mathbb{R}^{d\times d}$ be such that

$$|\mathbf{S}(c, \mathbf{M})| \leq C(|\operatorname{sym}(\mathbf{M})|^{q-1} + 1) \tag{1.10}$$

$$|\mathbf{S}(c_1, \mathbf{M}) - \mathbf{S}(c_2, \mathbf{M})| \leq C|c_1 - c_2|(|\operatorname{sym}(\mathbf{M})|^{q-1} + 1) \tag{1.11}$$

$$\mathbf{S}(c, \mathbf{M}) : \mathbf{M} \geq \omega|\operatorname{sym}(\mathbf{M})|^q - C_1 \tag{1.12}$$

for all $\mathbf{M} \in \mathbb{R}^{d\times d}$, $c, c_1, c_2 \in [a, b]$, and some $C, C_1, \omega > 0$, $q \in (\frac{2d}{d+2}, \infty)$. Here $\operatorname{sym}(\mathbf{M}) = \frac{1}{2}(\mathbf{M} + \mathbf{M}^T)$ and $\mathbf{A} : \mathbf{B} = \operatorname{tr}(\mathbf{A}^T\mathbf{B})$. Moreover, we assume that $\mathbf{S}(c, \cdot)\colon \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ is strictly monotone for every $c \in [a, b]$, where $\mathbb{R}^{d\times d}_{\mathrm{sym}} = \{A \in \mathbb{R}^{d\times d} : A^T = A\}$.

For the following we denote

$$E_{\mathrm{mix}}(c) = \int_\Omega \frac{|\nabla c|^2}{2}\,dx + \int_\Omega \Phi(c)\,dx.$$

Let $\mathbf{v} \in L^q(0, T; W^1_{q,0}(\Omega)^d) \cap L^\infty(0, T; L^2_\sigma(\Omega))$, $c \in L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega))$ with $\Phi(c) \in L^2(\Omega\times(0, T))$, and $\mu \in L^2(0, T; H^1(\Omega))$, where $0 < T < \infty$. Then $(\mathbf{v}, c, \mu)$ is a weak solution of the system (1.1)–(1.7) if for any $\boldsymbol{\varphi} \in C^\infty(\overline{Q_T})^d$ with $\operatorname{div}\boldsymbol{\varphi} = 0$ and $\operatorname{supp}(\boldsymbol{\varphi}) \subset\subset \Omega\times[0, T)$ the following holds:

$$\begin{aligned} -\int_{Q_T}\mathbf{v}\cdot\partial_t\boldsymbol{\varphi}\,d(x,t) - \int_{Q_T}\mathbf{v}\otimes\mathbf{v} : \mathbf{D}\boldsymbol{\varphi}\,d(x,t) + \int_{Q_T}\mathbf{S}(c, \mathbf{Dv}) : \mathbf{D}\boldsymbol{\varphi}\,d(x,t) \\ = \int_{Q_T}\nabla c\otimes\nabla c : \mathbf{D}\boldsymbol{\varphi}\,d(x,t) + \int_\Omega \mathbf{v}_0\cdot\boldsymbol{\varphi}(0)\,dx \end{aligned} \tag{1.13}$$