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Upper and lower bounds of time decay rate of solutions to a class of incompressible third grade fluid equations[☆]

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ABSTRACT

This paper discusses the large time behaviors of solutions for a class of incompressible third grade fluid equations in $\mathbb{R}^3$. Using the Fourier splitting method of Schonbek, the authors prove the upper and lower bounds of the time decay rate in $\mathbb{L}^2$ for the weak solutions. The upper and lower bounds of decay rate are optimal in the sense that they coincide with the upper and lower bounds of the decay rate of solutions to the heat equation.

1. Introduction

Fluids of different type form an important class of non-Newtonian fluids. The fluids of grade n, introduced by Rivlin and Ericksen [1], are the fluids for which the stress tensor is a polynomial of degree n in the first n Rivlin–Ericksen tensor defined recursively by

$$\begin{aligned} &A_1(u) = A(u) = \nabla u + (\nabla u)^T, \\ &A_{k+1}(u) = \frac{D}{Dt}A_k(u) + (\nabla u)^T A_k(u) + A_k(u)\nabla u, \quad k = 1, 2, \ldots, \end{aligned}$$

where $\frac{D}{Dt} = \frac{\partial}{\partial t} + u \times \nabla$ denotes the material derivative and $(\nabla u)^T$ is the transposition of the Jacobian matrix ∇u. In [1], the constitutive relation of a particular fluid of grade n is given by $T = -pI + F(A_1, A_2, \ldots, A_n)$, where I is the identity matrix of order n and F is an isotropic polynomial of degree n.

There are some references on the existence, uniqueness and asymptotic behaviors of solutions for second and third grade fluid equations, see e.g. [2–6] and the references therein. In [4], Busuioc and Iftimie studied the existence of solutions to the following third grade fluid equations

$$\begin{cases} \partial_t v + (u \cdot \nabla)v + \sum_j v_j \nabla u_j - \nu \Delta u = (\alpha_1 + \alpha_2)\mathrm{div}\big(A^2(u)\big) + \beta \mathrm{div}\big(|A(u)|^2 A(u)\big) - \nabla p, \\ v = u - \alpha_1 \Delta u, \\ \mathrm{div}\, u = 0, \\ u(x, 0) = u_0, \end{cases} \tag{1.1}$$

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in $\mathbb{R}^n (n = 2, 3)$, with the coefficients ν, α_1, α_2 and β satisfy the following hypotheses:

$$\nu \geqslant 0, \qquad \alpha_1 > 0, \qquad \beta \geqslant 0, \qquad |\alpha_1 + \alpha_2| \leqslant \sqrt{24\nu\beta}.$$

Busuioc and Iftimie in [4] proved that Eq. (1.1) possesses a global solution if the initial value is in $\mathbb{H}^2(\mathbb{R}^n)$ $(n = 2, 3)$. They also established the uniqueness of solutions for Eq. (1.1) when $n = 2$.

Recently, Hamza and Paicu studied in [5] a particular case of the third grade fluids Eq. (1.1) in $\mathbb{R}^3$, where they assumed $\alpha_1 = 0$. In this case, Eq. (1.1) become

$$\begin{cases} \partial_t u + (u \cdot \nabla)u - \nu\Delta u - \alpha \operatorname{div}\big(A^2(u)\big) - \beta \operatorname{div}\big(|A(u)|^2 A(u)\big) + \nabla p = 0, \\ \operatorname{div} u = 0, \\ u(x, 0) = u_0, \end{cases} \tag{1.2}$$

where we have denoted α_2 by α. With the following assumptions on the coefficients:

$$\beta > 0 \quad \text{and} \quad |\alpha| < \sqrt{2\nu\beta}.$$

Hamza and Paicu proved the existence and uniqueness, as well as the stability (when $|\alpha| < \sqrt{\nu\beta/2}$), of global weak solutions for Eq. (1.2) with natural regularity assumption on the initial data belonging to the energy space $\mathbb{L}^2(\mathbb{R}^3)$. They also proved that if the initial datum belongs to $\mathbb{H}^1(\mathbb{R}^3)$, then the solution belongs to $\mathbb{H}^1(\mathbb{R}^3)$ for any positive time and they gave a control of the $\mathbb{H}^1(\mathbb{R}^3)$ norm of the solution.

The goal of this paper is to investigate the time decay rate of weak solutions for Eq. (1.2) in $\mathbb{R}^3$. As far as we know, there are no references concerning this aspect for the third grade incompressible fluids. We will prove the upper and lower bounds of the decay rate in $\mathbb{L}^2$ for the weak solutions. Roughly speaking, if the initial value $u_0 \in \mathbb{L}^1(\mathbb{R}^3) \cap \mathbb{L}^2(\mathbb{R}^3)$ and $\operatorname{div} u_0 = 0$, then we can establish that the solution $u(x, t)$ satisfies the following upper bound of decay rate:

$$\|u(x, t)\| \leqslant c(1 + t)^{-3/4}, \quad t > 1. \tag{1.3}$$

If the initial value $u_0 \in \mathbb{L}^1(\mathbb{R}^3) \cap \mathbb{L}^2(\mathbb{R}^3) \cap \mathcal{R}_\kappa^\delta$ and $\operatorname{div} u_0 = 0$, then we can prove that the solution $u(x, t)$ satisfies the following lower bound of decay rate:

$$\|u(x, t)\| \geqslant c(1 + t)^{-3/4}, \quad \text{for large } t, \tag{1.4}$$

where κ and δ are some positive constants and

$$\mathcal{R}_\kappa^\delta := \{u \,|\, |\widehat{u}(\xi)| \geqslant \kappa \text{ for } |\xi| \leqslant \delta\} \tag{1.5}$$

is introduced by Schonbek [7] to describe the set of functions whose Fourier transform near the origin possesses a lower bound.

We also prove that the above upper and lower bounds of decay rate are optimal in the sense that they coincide with the upper and lower bounds of decay rate for the solutions of the following heat equations

$$\begin{cases} \partial_t v = \Delta v, \\ v(x, 0) = u_0. \end{cases} \tag{1.6}$$

Let v be a solution of Eq. (1.6). If $u_0(x) \in \mathbb{L}^1(\mathbb{R}^2) \cap \mathbb{L}^2(\mathbb{R}^2) \cap \mathcal{R}_\kappa^\delta$, then combining the estimates of Schonbek [7] and the $L^p - L^q$ estimate (see e.g. [8–10]), we have

$$c_1(1 + t)^{-3/4} \leqslant \|v(x, t)\| \leqslant \|u_0\|_{\mathbb{L}^1(\mathbb{R}^3)}(1 + t)^{-3/4}, \quad t > 0, \tag{1.7}$$

where c_1 is a positive constant depending on κ and δ. In fact, we can prove that

$$\|u(x, t) - v(x, t)\| \leqslant c(1 + t)^{-1}, \quad t > 1, \tag{1.8}$$

if the initial value $u_0 \in \mathbb{L}^1(\mathbb{R}^3) \cap \mathbb{L}^2(\mathbb{R}^3)$ and $\operatorname{div} u_0 = 0$.

We should point out that the proof of this paper is greatly inspired from the work of Schonbek [7,11] for the Navier–Stokes equations. In [7,11], Schonbek used the Fourier splitting method and some delicate analyses to prove the upper and lower bounds of decay in $\mathbb{L}^2$ for the Leray–Hopf solutions of the Navier–Stokes equations. Later, this Fourier splitting method was well extended and was combined with the $L^p - L^q$ estimate (see e.g. [8–10]) to investigate the decay for the solutions of partial differential equations from mathematical physics.

Nowadays, there are many works discussing the decay rate of solutions for nonlinear evolution equations. For example, one can see Wiegner [12], Schonbek [13,14], Schonbek and Wiegner [15], Zhang [16,17], Oliver and Titi [18], He and Xin [19], Brandolese and Vigneron [20], Dong et al. [21,22], Han [23,24] for the Navier–Stokes equations; see Bae [25], Guo and Zhu [26], Nečasová and Penel [27], Dong et al. [28,29] for the non-Newtonian fluid equations; see Liu [30], Wang et al. [31], Li et al. [32], Brandolese and Schonbek [33] for the Boussinesq equations; see Wang and Yu [34], Dai et al. [35] for the liquid crystals systems; see Schonbek et al. [36,37] for the MHD equations; see Niche and Schonbek [38] for the quasi-geostrophic equations, and see Guo and Wang [39] for the heat equation, Navier–Stokes equations and Boltzmann equation, etc.