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Gevrey regularity for solutions of the non-cutoff Boltzmann equation: The spatially inhomogeneous case

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ABSTRACT

In this paper we consider the non-cutoff Boltzmann equation in the spatially inhomogeneous case. We prove the propagation of Gevrey regularity for the so-called smooth Maxwellian decay solutions to the Cauchy problem of spatially inhomogeneous Boltzmann equation, and obtain Gevrey regularity of order $1/(2s)$ in the velocity variable v and order 1 in the space variable x. The strategy relies on our recent results for the spatially homogeneous case [T.-F. Zhang and Z. Yin, Gevrey regularity of spatially homogeneous Boltzmann equation without cutoff, J. Differential Equations 253 (4) (2012), 1172–1190. http://dx.doi.org/10.1016/j.jde.2012.04.023]. Rather, we need much more intricate analysis additionally in order to handle with the coupling of the double variables. Combining with the previous result mentioned above, it gives a characterization of the Gevrey regularity of the particular kind of solutions to the non-cutoff Boltzmann.

1. Introduction

1.1. The Boltzmann equation

In this paper we consider the Cauchy problem of the spatially inhomogeneous Boltzmann equation without angular cutoff. It reads, with a $T > 0$, as the following equation,

$$\begin{cases} f_t(t,x,v) + v\cdot\nabla_x f(t,x,v) = Q(f,f)(v), & t\in(0,T], \\ f(0,x,v) = f_0(x,v), \end{cases} \tag{1.1}$$

for the density distribution function of particles $f = f(t,x,v)$, which are located around position $x \in \mathbb{T}^3$ with velocity $v \in \mathbb{R}^3$ at time $t \geq 0$. The right-hand side of the above equation is the so-called Boltzmann bilinear collision operator acting only on the velocity variable v:

$$Q(g,f) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-v_*,\sigma)\left\{g'_*f' - g_*f\right\} d\sigma dv_*.$$

Above, we use the standard shorthand $f = f(t,x,v)$, $f_* = f(t,x,v_*)$, $f' = f(t,x,v')$, $f'_* = f(t,x,v'_*)$. The relations between the post- and pre-collisional velocities are described by the σ-representation, that is, for $\sigma \in \mathbb{S}^2$,

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma, \qquad v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma.$$

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Note that the collision process satisfies the conservation of momentum and kinetic energy, i.e.

$$v + v_* = v' + v'_*, \qquad |v|^2 + |v_*|^2 = |v'|^2 + |v'_*|^2.$$

The collision cross section $B(z, \sigma)$ is a given non-negative function depending only on the interaction law between particles. From a mathematical viewpoint, that is to say, $B(z, \sigma)$ depends only on the relative velocity $|z| = |v - v_*|$ and the deviation angle θ defined through the scalar product $\cos\theta = \frac{z}{|z|} \cdot \sigma$.

Without loss of generality, the cross section B is assumed to be of the form

$$B(v - v_*, \cos\theta) = \Phi(|v - v_*|)b(\cos\theta), \quad \cos\theta = \frac{v - v_*}{|v - v_*|} \cdot \sigma, 0 \le \theta \le \frac{\pi}{2},$$

where the kinetic factor Φ is given by

$$\Phi(|v - v_*|) = |v - v_*|^{\gamma},$$

and the angular part b, with a singularity, satisfies

$$\sin\theta b(\cos\theta) \sim \theta^{-1-2s}, \quad \text{as } \theta \to 0+,$$

for some $0 < s < 1$.

We remark that if the inter-molecule potential is given by the inverse-power law $U(\rho) = \rho^{-(p-1)}$ (where $p > 2$), it holds that $\gamma = \frac{p-5}{p-1}$ and $s = \frac{1}{p-1}$. Generally, the cases $\gamma > 0$, $\gamma = 0$, and $\gamma < 0$ correspond to the so-called hard, Maxwellian, and soft potential respectively. And the cases $0 < s < 1/2$, $1/2 \le s < 1$ correspond to the so-called mild singularity and strong singularity respectively.

1.2. Review of non-cutoff theory in Gevrey spaces

We begin with a brief review for the non-cutoff theory of the Boltzmann equation. We refer to Villani's review book [1] for the physical background and the mathematical theories of the Boltzmann equation. Furthermore, in the non-cutoff setting, Alexandre gave more details in [2].

Our discussion is based on the following definition of Gevrey spaces $G^s(\Omega)$ on an open subset $\Omega \subseteqq \mathbb{R}^3$ (see [3], for instance).

Definition 1.1. For $0 < s < +\infty$, we say that $f \in G^s(\Omega)$, if $f \in C^\infty(\Omega)$, and there exist $C > 0$, $N_0 > 0$ such that

$$\|\partial^\alpha f\|_{L^2(\Omega)} \le C^{|\alpha|+1}\{\alpha!\}^s, \quad \forall \alpha \in \mathbb{N}^3, |\alpha| \ge N_0.$$

Note that the Gevrey scale measures regularity between analytic and C^∞. More precisely, when $s = 1$, it is usual analytic function. If $s > 1$, it is a Gevrey class function. And for $0 < s < 1$, it is called an ultra-analytic function.

For the Cauchy problem of the Boltzmann equation in Gevrey classes, Ukai showed, in [4] in 1984, that there exists a unique local solution for both spatially homogeneous and inhomogeneous cases, with the assumption on the cross section:

$$\left|B(|z|, \cos\theta)\right| \le K(1 + |z|^{-\gamma'} + |z|^{\gamma})\theta^{-n+1-2s}, \quad n \text{ is dimensionality},$$
$$(0 \le \gamma' < n, 0 \le \gamma < 2, 0 \le s < 1/2, \gamma + 6s < 2).$$

In particular, for the spatially inhomogeneous case, by introducing the norm of Gevrey space

$$\|f\|_{\delta,\rho_1,\nu_1,\rho_2,\nu_2} = \sum_{\alpha,\beta} \frac{\rho_1^{|\alpha|}\rho_2^{|\beta|}}{\{\alpha!\}^{\nu_1}\{\beta!\}^{\nu_2}} \|e^{\delta\langle v\rangle^2}\partial_x^\alpha\partial_v^\beta f\|_{L^\infty(\mathbb{R}_x^n\times\mathbb{R}_v^n)}.$$

Ukai proved that, under some assumptions for ν and the initial datum $f_0(x, v)$, the Cauchy problem (1.1) has a unique solution $f(t, x, v)$ for $t \in (0, T]$.

On the other hand, Desvillettes established in [5] the C^∞ smoothing effect for solutions of the Cauchy problem in the spatially homogeneous case, and conjectured the Gevrey smoothing effect. He also proved, without any assumptions on the decay at infinity in v variables, the propagation of Gevrey regularity for solutions (see [6]).

In 2009 Morimoto et al. considered in [7] the Gevrey regularity for the linearized Boltzmann equation around the absolute Maxwellian distribution, by virtue of the following mollifier:

$$G_\delta(t, D_v) = \frac{e^{t\langle D_v\rangle^{1/\nu}}}{1 + \delta e^{t\langle D_v\rangle^{1/\nu}}}, \quad 0 < \delta < 1.$$

We remark that the same operator was used in many related models such as the Fokker–Planck equation, Kac's equation, the Landau equation, and so on.