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Solving inverse problems for differential equations by a "generalized collage" method and application to a mean field stochastic model

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ABSTRACT

In the first part of this paper, after recalling how to solve inverse problems for deterministic and random differential equations using the collage method, we switch to the analysis of stochastic differential equations. Here inverse problems can be solved by minimizing the collage distance in an appropriate metric space. In the second part, we develop a general collage coding framework for inverse problems for boundary value problems. Although a general inverse problem can be very complicated, via the Generalized Collage Theorem presented in this paper, many such problems can be reduced to an optimization problem which can be solved at least approximately. We recall some previous results by some of the authors on the same topic, but we provide more numerical examples to analyze the stability of the generalized collage method under perturbation of data. We then extend these results to the case of diffusion equations. Finally, we show an application of this methodology to a system of coupled stochastic differential equations which describes the interaction between particles in a physical system.

1. Introduction

Many inverse problems may be viewed in terms of the approximation of a target element u in a complete metric space (X, d) by the fixed point $\bar{u}$ of a contraction mapping $T : X \to X$. In practical applications, from a family of contraction mappings T_λ, $\lambda \in \Lambda \subset \mathbb{R}^n$, one wishes to find the parameter $\bar{\lambda}$ for which the approximation error $d(u, \bar{u}_\lambda)$ is as small as possible. Thanks to a simple consequence of Banach's fixed point theorem known as the "Collage Theorem", most practical methods of solving the inverse problem for fixed point equations seek to find an operator T for which the *collage distance* $d(u, Tu)$ is as small as possible.

Theorem 1.1 (*"Collage Theorem" [1]*). *Let (X, d) be a complete metric space and $T : X \to X$ a contraction mapping with contraction factor $c \in [0, 1)$. Then for any $u \in X$,*

$$d(u, \bar{u}) \le \frac{1}{1-c} d(u, Tu), \tag{1}$$

where $\bar{u}$ is the fixed point of T.

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One now seeks a contraction mapping T that minimizes the so-called *collage error* $d(u, Tu)$—in other words, a mapping that sends the target u as close as possible to itself. This is the essence of the method of *collage coding* which has been the basis of most, if not all, fractal image coding and compression methods.

In [2] (and subsequent works [3–9]), the authors showed how collage coding could be used to solve inverse problems for systems of differential equations having the form

$$\begin{cases} \dot{u} = f(t, u), \\ u(0) = u_0, \end{cases} \tag{2}$$

when f is a polynomial and by reducing the problem to the corresponding Picard integral operator associated with it,

$$(Tu)(t) = u_0 + \int_0^t f(s, u(s))\,\mathrm{d}s. \tag{3}$$

Here we show how one can attack this problem in the general case when f belongs to L^2. Let us consider the complete metric space $C([0, T])$ endowed with the usual d_∞ metric and assume that $f(t, x)$ is Lipschitz in the variable x, that is there exists a $K \geq 0$ such that $|f(s, x_1) - f(s, x_2)| \leq K|x_1 - x_2|$, for all $x_1, x_2 \in \mathbb{R}$. For simplicity we suppose that $x \in \mathbb{R}$ but the same consideration can be developed for the case of several variables. Under these hypotheses T is Lipschitz on the space $C([-\delta, \delta] \times [-M, M])$ for some δ and $M > 0$.

Theorem 1.2 ([2]). *The function T satisfies*

$$\|Tu - Tv\|_2 \leq c\|u - v\|_2 \tag{4}$$

for all $u, v \in C([-\delta, \delta] \times [-M, M])$ *where* $c = \delta K$.

Now let $\delta' > 0$ be such that $\delta' K < 1$. In order to solve the inverse problem for (3) we take the L^2 expansion of the function f. Let $\{\phi_i\}$ be a basis of functions in $L^2([-\delta', \delta'] \times [-M, M])$ and consider

$$f_\lambda(s, x) = \sum_{i=1}^{+\infty} \lambda_i \phi_i(s, x). \tag{5}$$

Each sequence of coefficients $\lambda = \{\lambda_i\}_{i=1}^{+\infty}$ then defines a Picard operator T_λ. Suppose further that each function $\phi_i(s, x)$ is Lipschitz in x with constants K_i.

Theorem 1.3 ([2]). *Let* $K, \lambda \in \ell^2(\mathbb{R})$. *Then*

$$|f_\lambda(s, x_1) - f_\lambda(s, x_2)| \leq \|K\|_2 \|\lambda\|_2 |x_1 - x_2| \tag{6}$$

for all $s \in [-\delta', \delta']$ *and* $x_1, x_2 \in [-M, M]$ *where* $\|K\|_2 = \left(\sum_{i=1}^{+\infty} K_i^2\right)^{\frac{1}{2}}$ *and* $\|\lambda\|_2 = \left(\sum_{i=1}^{+\infty} \lambda_i^2\right)^{\frac{1}{2}}$.

Given a target solution x, we now seek to minimize the collage distance $\|u - T_\lambda u\|_2$. The square of the collage distance becomes

$$\begin{aligned} \Delta^2(\lambda) &= \|u - T_\lambda u\|_2^2 \\ &= \int_{-\delta}^{\delta} \left| u(t) - \int_0^t \sum_{i=1}^{+\infty} \lambda_i \phi_i(s, u(s))\mathrm{d}s \right|^2 \mathrm{d}t \end{aligned} \tag{7}$$

and the inverse problem can be formulated as

$$\min_{\lambda \in \Lambda} \Delta(\lambda), \tag{8}$$

where $\Lambda = \{\lambda \in \ell^2(\mathbb{R}) : \|\lambda\|_2 \|K\|_2 < 1\}$. To solve numerically this problem, let us consider the first n terms of the L^2 basis; in this case the previous problem can be reduced to:

$$\min_{\lambda \in \tilde{\Lambda}} \tilde{\Delta}^2(\lambda) = \int_{-\delta}^{\delta} \left| x(t) - \int_0^t \sum_{i=1}^{n} \lambda_i \phi_i(s, x(s))\mathrm{d}s \right|^2 \mathrm{d}t, \tag{9}$$

where $\tilde{\Lambda} = \{\lambda \in \mathbb{R}^n : \|\lambda\|_2 \|K\|_2 < 1\}$. This is a classical quadratic optimization problem which can be solved by means of classical numerical methods. Let $\tilde{\Delta}^n_{\min}$ be the minimum value of $\tilde{\Delta}$ over $\tilde{\Lambda}$. This is a non-increasing sequence of numbers (depending on n) and as shown in [10] it is possible to show that $\liminf_{n\to+\infty} \tilde{\Delta}^n_{\min} = 0$. This states that the distance between the target element and the unknown solution of the differential equation can be made arbitrary small.