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Homogenization of a variational inequality for the Laplace operator with nonlinear restriction for the flux on the interior boundary of a perforated domain

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ABSTRACT

In this paper, we study the asymptotic behavior of solutions u_ε of the elliptic variational inequality for the Laplace operator in domains periodically perforated by balls with radius of size $C_0\varepsilon^\alpha$, $C_0 > 0$, $\alpha \in (1, \frac{n}{n-2}]$, and distributed with period ε. On the boundary of the balls, we have the following nonlinear restrictions $u_\varepsilon \geq 0$, $\partial_\nu u_\varepsilon \geq -\varepsilon^{-\gamma}\sigma(x, u_\varepsilon)$, $u_\varepsilon\left(\partial_\nu u_\varepsilon + \varepsilon^{-\gamma}\sigma(x, u_\varepsilon)\right) = 0$, $\gamma = \alpha(n-1) - n$. The weak convergence of the solutions u_ε to the solution of an effective problem is given. In the critical case $\alpha = \frac{n}{n-2}$, the effective equation contains a nonlinear term which has to be determined as a solution of a functional equation. Furthermore, a corrector result with respect to the energy norm is proved.

1. Introduction

In this paper, we are considering variational inequalities arising e.g., in modeling diffusion of substances in a domain with inclusions. It is assumed that nonlinear adsorption is taking place at the boundary of these inclusions. Here, we are interested in the case where the number of inclusions is large, their distribution is periodical of period ε and the size of each is very small of order ε^α. In the literature, these perforations are called *small* or *tiny holes*. On the other hand, we also suppose that the process on the boundary is of order $\varepsilon^{-\gamma}$, that means, we assume strong processes on the small inclusions.

Passing to the scale limit, effective equations are derived for the considered variational inequalities. Hereby, we consider the range for the parameters α, and γ for which the adsorption process on the inclusion at the micro-scale gives rise to an effective sink/source term in the macroscopic equation. It turns out that we have to distinguish between two cases. In the first case, when $\alpha \in \left(1, \frac{n}{n-2}\right)$, and $\gamma = \alpha(n-1) - n$, the nonlinearity in the sink/source term has the same form as the nonlinearity in the boundary condition of the ε-problem. However, the second case, when $\alpha = \frac{n}{n-2} = \gamma$, is the more interesting one. In this case, the nonlinearity in the sink/source term appearing in the effective equation has a different form from the nonlinearity in the ε-problem, and has to be determined as a solution of a functional equation.

We are assuming balls as inclusions and we concentrate on developing methods needed for the derivation of the effective model. More general inclusions are covered in a following investigation. The progress achieved in this paper consists in treating variational inequalities for the considered nonlinear restrictions, and discussing different scalings in geometry and

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processes. More precisely, we study the asymptotic behavior of solutions u_ε of the elliptic variational inequality in domains periodically perforated by balls with radius of size

$$C_0\varepsilon^\alpha, \quad C_0 > 0,\ \alpha \in \left(1, \frac{n}{n-2}\right], \tag{1}$$

which are distributed with period ε in a domain Ω. On the boundary of the balls, we have the following restrictions

$$u_\varepsilon \geq 0, \qquad \partial_\nu u_\varepsilon \geq -\varepsilon^{-\gamma}\sigma(x, u_\varepsilon), \quad u_\varepsilon\Big(\partial_\nu u_\varepsilon + \varepsilon^{-\gamma}\sigma(x, u_\varepsilon)\Big) = 0 \tag{2}$$

with $\gamma = \alpha(n-1) - n$. We prove weak convergence of the solutions $\{u_\varepsilon\}$ as $\varepsilon \to 0$ to the solution of the homogenized problem. Furthermore, we give corrector results with respect to the energy norm.

The literature on problems formulated in domains with small holes is very wide, and started with the works [1,2] (see also [3]), where the Poisson equation with homogeneous boundary conditions in a domain with small holes was considered. The corresponding problems for Neumann boundary conditions were treated e.g. in [4,5]. In [6–9] Poisson problems in domains with small holes and mixed boundary conditions on the boundary of the inclusions were treated. Nonlinear problems in domains with small holes were treated among others in [10–13]. Homogenization problems for variational inequalities were considered in [14–19]. A recent development concerning the techniques for problems in domains with small holes is given e.g. in [20,21] (see also the references therein), where the unfolding method is used to treat problems in perforated domains. Concerning the problems in domains with holes of diameter ε, there is a huge literature available, starting with the paper [22]. We mention here a few of them, like e.g. [23–31].

There are different methods used to derive homogenized (effective) models in the case of domains with small holes. Many of them consist in constructing suitable extension operators (from the perforated, ε-dependent domains to a fixed domain) and choosing test functions adapted to the structure of the underlying problem. An alternative approach is the unfolding method, where the extension operators are not necessary. However, until now, this method was applied mainly to linear problems.

The problem considered in the present paper, a variational inequality for the Laplace operator with nonlinear third type boundary conditions, is a generalization of the problems treated in [10,11], where the corresponding equation with nonlinear third type boundary condition was considered. In [10], the problem was treated by introducing auxiliary elliptic boundary value problems, and the corresponding minimizing problems. Then the convergence of the sequence of minimizers was shown. In [11], a different, more direct approach was developed. There test functions were introduced directly for the non-stationary problem; a basic ingredient for these test functions were the test functions w^ε from [2]. In [6], where the case of linear third type boundary conditions on the small holes was considered, epi-convergence methods for sequences of functionals are used.

In the actual paper, we use a similar approach to [11]. We further generalize the test functions in order to deal with nonlinear inequalities.

2. Setting of the problem

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 3$, with a smooth boundary $\partial\Omega$ and $Y = (-1/2, 1/2)^n$. We denote by G_0 the ball of radius 1 with its center in the origin of coordinates. For a set B, and $\delta > 0$, we denote by δB the set $\{x \mid \delta^{-1}x \in B\}$. Let $\varepsilon > 0$ be a small, positive parameter, and set $\widetilde{\Omega}_\varepsilon = \{x \in \Omega \mid \rho(x, \partial\Omega) > 2\varepsilon\}$. For $a_\varepsilon = C_0\varepsilon^\alpha$, where $\alpha \in (1, n/(n-2)]$, and C_0 is a positive constant, we define

$$G_\varepsilon = \bigcup_{j\in\Upsilon_\varepsilon}(a_\varepsilon G_0 + \varepsilon j) = \bigcup_{j\in\Upsilon_\varepsilon} G_\varepsilon^j,$$

where $\Upsilon_\varepsilon = \{j \in \mathbb{Z}^n : (a_\varepsilon G_0 + \varepsilon j) \cap \overline{\widetilde{\Omega}}_\varepsilon \neq \emptyset\}$, and $|\Upsilon_\varepsilon| \cong d\varepsilon^{-n}$, $d = \text{const} > 0$, $\mathbb{Z}^n$ is the set of vectors z with integer components. Furthermore, let $Y_\varepsilon^j = \varepsilon Y + \varepsilon j$, and note that $\overline{G}_\varepsilon^j \subset \overline{Y}_\varepsilon^j$ and the center of G_ε^j coincides with the center of the cube Y_ε^j. We set

$$\Omega_\varepsilon = \Omega \setminus \overline{G_\varepsilon}, \qquad S_\varepsilon = \partial G_\varepsilon, \qquad \partial\Omega_\varepsilon = \partial\Omega \cup S_\varepsilon.$$

In Ω_ε we consider the following problem: Find $u_\varepsilon \in K_\varepsilon$, such that the following variational inequality is satisfied for all $v \in K_\varepsilon$:

$$\int_{\Omega_\varepsilon} \nabla u_\varepsilon \nabla(v - u_\varepsilon)dx + \varepsilon^{-\gamma}\int_{S_\varepsilon}\sigma(x, u_\varepsilon)(v - u_\varepsilon)ds \geq \int_{\Omega_\varepsilon} f(v - u_\varepsilon)dx. \tag{3}$$

Here, the set K_ε is defined by

$$K_\varepsilon = \{g \in H^1(\Omega_\varepsilon, \partial\Omega) : g \geq 0 \text{ a.e. on } S_\varepsilon\}, \tag{4}$$