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A note on analysis and numerics of algae growth

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ABSTRACT

We extend the mathematical model for algae growth as described in Pham Thi (2006) [7] to include new effects. The roles of light, nutrients and acidity of the water body are taken into account. Important properties of the model such as existence and uniqueness of solution, as well as boundedness and positivity are investigated. We also discuss the numerical integration of the resulting system of ordinary differential equations and derive a condition which guarantees positivity of the numerical solution. The behavior of the model is demonstrated by numerical experiments.

1. Introduction

Existing models for algae growth can be categorized into two groups [1]. The first group, of "hard models", consists of models derived from first principles and heuristic assumptions without relying on validation with experimental data. The models in the second group are referred to as "soft models" and are designed by fitting the parameters of the model to real data. A prominent representative of the first group is the model proposed by Huisman and Weissing [2]. It was one of the first to recognize the role of depth-dependence of light intensity on the dynamics of algae population. In their work, the algae concentration is averaged across the vertical direction. This made the model simple but also less realistic. A later paper [3] extended this work to a full three-dimensional model which could account for mixing in all directions.

Malve et al. [4] and Haario et al. [5] applied the "soft modeling" approach to the analysis of an algae growth problem. The algae dynamics was modeled as a nonlinear system of ordinary differential equations (ODEs) with parameters fitted to data consisting of eight years of observations. The model incorporates reaction rate dependence on the temperature, but neglects the effects of light attenuation with depth.

In our model we consider the algae concentration to be depth-dependent, leading to a partial differential equation in space and time. The main factors which influence the amount of algae biomass B are the concentrations of phosphates P, nitrates N, and carbon dioxide C in the water as well as the intensity of light.

$$B + N + P + C \xrightarrow{light} B.$$

The light intensity has no direct effect on the the nutrients (N and P) and the carbon dioxide, which are also assumed to have a large diffusion rate. This implies that their concentration may be considered space-independent and modeled by a set of ODEs. We incorporate the ODEs (fitted to experimental data) from [5] into our model. This allows realistic reaction rates and reliable output.

With regard to the above classification of algae growth models, the proposed model is a hybrid one: it contains a hard PDE model for the biomass concentration and a soft ODE model for the nutrients and carbon dioxide. This model offers insight on the influence of mixing and of the pH on the growth rate. The pH of the medium can be controlled by adding carbon dioxide, which may be taken as an input parameter in an optimization problem.

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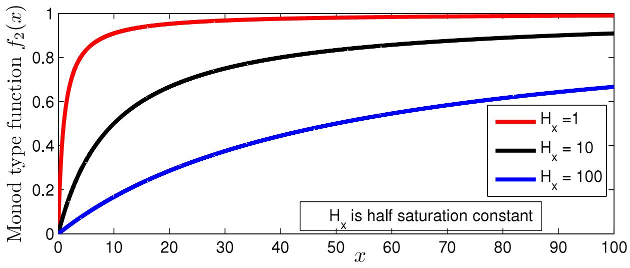

Fig. 1. Monod-type function for different half-saturation constants.

This work contributes to present scientific literature in the following ways: in terms of modeling, we extend the model in [2,3] to include the effect of pH on the algae growth rate. Further, we investigate the well-posedness for the time-dependent model presented here. In terms of numerics, we have derived the condition on the time step to ensure the positivity of the solution for the implicit 2-step BDF2 method used here which is a novelty. We perform numerical computations to study the time-dependent behavior of the growth rates of algae and depletion of minerals.

The paper is organized as follows. In Section 2 the mathematical model is presented. Section 3 is concerned with the analysis of the model. A numerical scheme is proposed in Section 4 together with conditions which guarantee positivity of the numerical solution. Section 5 reports on simulations which demonstrate the behavior of the model. Finally, our conclusions are presented in Section 6.

2. Mathematical model

We model the growth of the algae (biomass) in the water body. The biomass growth rate is related to the process of photosynthesis, the process of mixing, and the death rate. The process of photosynthesis depends on the concentration of the nutrients, the availability of carbon dioxide and the availability of light. The death rate includes both the harvesting rate as well as the natural death rate of the algae. Since the light intensity is uneven at different depths of the water, it is important to stir the water causing the mixing of the algae. In our model advection is assumed to be absent.

In the horizontal plane, we consider no variation and hence, the growth rate is independent of x and y coordinates. Thus, it is sufficient to consider a one-dimensional domain denoted by $\Omega = [0, z_{\max}]$, where $z_{\max}$ is the depth of the water body.

In our model the concentration of biomass w of the algae biomass has the following growth rate

$$\partial_t w = g(I_{\text{in}})f_1(P)f_2(N)f_3(C)w + D_M\partial_{zz}w - H_a(w, C), \tag{1}$$

where I_{in}, P, N and C are respectively the light intensity, the concentrations of phosphorus, nitrogen and carbon dioxide. The mixing is modeled by a diffusion term with a constant coefficient D_M. Inclusion of the mixing term helps to understand the effect of mixing on the overall production rate of the algae. The functions $g(I_{\text{in}}), f_1(P), f_2(N)$ and $f_3(C)$ define the dependence of the biomass growth rate on the light intensity, the concentration of nutrients (phosphates and nitrates) and the carbon dioxide. The function H_a models the depletion rate of algae biomass and includes both the harvesting and the natural decay. A similar model is used in [6,7]. Following [8] the effect of light intensity on the biomass growth is modeled by a Monod function (see [9,3]),

$$g(I_{\text{in}}) = \frac{\mu_0 I_{\text{in}}}{H_L + I_{\text{in}}}, \tag{2}$$

where I_{in} is the effective light intensity received by the algae and H_L is the half-saturation intensity. The Monod form ensures that the growth rate is almost linear when the light intensity is very small, and that the growth rate remains bounded by μ_0 when I_{in} becomes very large. Fig. 1 illustrates a family of Monod functions for different half-saturation constants. The light intensity received by the algae is not uniform throughout the water body. The light intensity is attenuated by two factors: the presence of algae and the water surface. The presence of the algae in the top layers causes reduction in the available light for the algae in the deeper layers. Moreover, the water layers themselves cause attenuation in the available light intensity for the deeper layers. Considering the above discussions, the light intensity can be modeled by

$$I_{\text{in}}(w, z, t) = I_0(t)e^{-kz}e^{\left(-r_s\int_0^z w(s,t)\,ds\right)}, \tag{3}$$

where $I_0(t)$ is the incident light intensity which changes in time (for instance, day and night cycle). The above relation is also known as the Beer–Lambert law. The constant k is the specific light attenuation coefficient due to the water layer and r_s is the specific light attenuation coefficient due to the presence of algae.

For the nutrients (N and P), we once again take Monod-type rates

$$f_1(P) = \frac{k_P[P - P_c]_+}{H_P + [P - P_c]_+}, \tag{4}$$

$$f_2(N) = \frac{k_N[N - N_c]_+}{H_N + [N - N_c]_+}. \tag{5}$$