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The resolvent technique for investigating the stability of plane Couette flow: Bounds for the pressure terms

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ABSTRACT

We derive bounds for the pressure terms of the operator used to prove nonlinear stability of plane Couette flow through the resolvent technique.

1. Introduction

We discuss the stability of plane Couette flow via the resolvent method. Through this method, one can derive lower bounds for the norms of perturbations of the flow that can lead to turbulence. Our aim is to discuss and clarify a point that has been overlooked so far, which is determining how much control over the perturbations one should assume to derive a stability result via the resolvent technique. This is directly related to the integral part of an operator related to the flow. We discuss the two-dimensional case, but all the considerations can be used for the three-dimensional case with minor technical changes. The main difference between the cases of two and three spatial dimensions is that different resolvent estimates hold for the two cases, leading to different thresholds. This point will be made clear later on. We begin by describing the problem and discussing previous works using the resolvent method.

2. The problem

We are interested in the following initial–boundary value problem:

$$\begin{cases} u_t + (u \cdot \nabla)u + \nabla p = \dfrac{1}{R}\Delta u, \\ \nabla \cdot u = 0, \\ u(x, 0, t) = (0, 0), \\ u(x, 1, t) = (1, 0), \\ u(x, y, t) = u(x + 1, y, t), \\ u(x, y, 0) = f(x, y), \end{cases} \tag{1}$$

where $u : \mathbb{R} \times [0, 1] \times [0, \infty) \longrightarrow \mathbb{R}^2$ is the unknown function $u(x, y, t) = (u_1(x, y, t), u_2(x, y, t))$. The positive parameter R is the Reynolds number. The initial condition $f(x, y)$ is assumed to be smooth, divergence free and compatible with the

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boundary conditions. The pressure $p(x, y, t)$ can be determined in terms of u through the elliptic problem

$$\begin{cases} \Delta p = -\nabla \cdot ((u \cdot \nabla) u), \\ p_y(x, 0, t) = \dfrac{1}{R} u_{2yy}(x, 0, t), \\ p_y(x, 1, t) = \dfrac{1}{R} u_{2yy}(x, 1, t). \end{cases} \tag{2}$$

It can be easily seen that $U(x, y) = (y, 0)$, $P = constant$ is a steady solution of problem (1). The vector field $U(x, y) = (y, 0)$ is known as Couette flow.

Using the resolvent technique, one can prove and quantify asymptotic stability for this flow. By quantification we mean the derivation a number $M(R)$ such that disturbances of the flow with norm less than $M(R)$ will tend to zero as time t tends to infinity—in other words, deriving a lower bound for the norm of perturbations that can lead to turbulence. For a general discussion about the resolvent technique, see [1,2].

This problem has been studied for the case of three spatial dimensions in [3], and a threshold amplitude for perturbations was found to be of order $\mathcal{O}(R^{-\frac{21}{4}})$. The estimates for the resolvent of the linearized equations governing perturbations were those found in [4–7], predicting the resolvent constant of the linear operator associated with the problem to be proportional do R^2. In [8], the resolvent technique was used again to prove the stability of the three-dimensional problem, but the estimates for the resolvent constant were those in [9]. By using modified norms, the authors achieve $M(R)$ of order $\mathcal{O}(R^{-3})$ for two of the components of the perturbation, and of order $\mathcal{O}(R^{-4})$ for the remaining component. The analysis in [5] clarifies the reason for the R^2 growth of the L^2 norm of the resolvent, since it shows exactly where the extra factor of R comes into the game. Moreover, it also shows that different components of perturbations of the base flow have different scales with respect to R.

Here, we use the resolvent technique, and the same norms as [3], with the obvious modifications for the case of two spatial dimensions, with the main objective of clarifying a subtle point that has been overlooked in some works. In [3], it was stated that one needs control over the Sobolev norm H^2 of the perturbation to assure stability. Later on, in [8], the authors note that the H^2 norm is not enough, and claim that one needs control over the norm H^4. Actually, this is still not enough, since in one of the directions, one needs control over six derivatives of the perturbation. This necessity is due to the pressure terms appearing in the problem. In Section 6, we show in detail estimates for these terms, and clarify the reason for this requirement. Moreover, our argument shows that derivatives of different orders of the perturbation scale differently with the Reynolds number. In other words, to assure decay of the perturbations via the resolvent method, one should require the perturbation to be small in some weighted norm involving six derivatives, where the weights depend on the Reynolds number R. We carry out the whole proof of stability again just for the sake of clarity. For the two-dimensional case here, the resolvent method leads to a threshold amplitude of order $\mathcal{O}(R^{-3})$. We note that our argument is the same as that used in [3], with some minor differences (see also [10] for a detailed expository work). The only reason for the better exponent in our case is the better dependence of the resolvent constant on R for the two-dimensional case. In this case, the resolvent constant is proportional to R, as found in [11].

For more references about Couette flow related problems see, e.g., [12–23]. In particular, in the very nice work [20] the authors explain the so called Sommerfeld paradox: even though plane Couette flow is linearly stable for all Reynolds numbers, experimentally arbitrarily small perturbations can lead to turbulence when the Reynolds number is large enough.

This work is divided into four sections. In Section 3, we introduce some basic notation and derive the equations for perturbations of the Couette flow. In Section 4, we derive estimates for the solution of the linearized equations for the perturbations. In Section 5, we use those estimates to prove asymptotic stability for the flow, and to derive the threshold amplitude $M(R)$. In Section 6, we carefully derive the estimates for the pressure terms involved in the problem.

3. Preliminaries

We denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ the L_2 inner product and norm over $\Omega = [0, 1] \times [0, 1]$:

$$\langle u, w \rangle = \int_\Omega \overline{u} \cdot w dxdy; \qquad \|u\|^2 = \langle u, u \rangle.$$

All the matrix norms that appear in this paper are the usual Frobenius norms. The usual Sobolev norm of u over Ω is denoted by

$$\|u\|^2_{H^n(\Omega)} = \sum_{j=0}^{n} \|D^j u\|^2$$

where D^j denotes the jth derivative of u with respect to the space variables. Unless stated otherwise, all norms in the space variables will be calculated over Ω and therefore we will write $\| \cdot \|_{H^n(\Omega)}$ as $\| \cdot \|_{H^n}$. We make use of a two-dimensional version of the weighted norm $\| \cdot \|_{\widetilde{H}}$ used in [3]:

$$\|u\|^2_{\widetilde{H}} = \|u\|^2 + \frac{1}{R}\|Du\|^2 + \frac{1}{R^2}\|u_{xy}\|^2. \tag{3}$$