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Stationary distributions of semistochastic processes with disturbances at random times and with random severity

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ABSTRACT

We consider a semistochastic continuous-time continuous-state space random process that undergoes downward disturbances with random severity occurring at random times. Between two consecutive disturbances, the evolution is deterministic, given by an autonomous ordinary differential equation. The times of occurrence of the disturbances are distributed according to a general renewal process. At each disturbance, the process gets multiplied by a continuous random variable ("severity") supported on $[0, 1)$. The inter-disturbance time intervals and the severities are assumed to be independent random variables that also do not depend on the history.

We derive an explicit expression for the conditional density connecting two consecutive post-disturbance levels, and an integral equation for the stationary distribution of the post-disturbance levels. We obtain an explicit expression for the stationary distribution of the random process. Several concrete examples are considered to illustrate the methods for solving the integral equations that occur.

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1. Introduction and set-up of the problem

Random disturbances of physical, chemical and biological systems occur commonly. The effects of such phenomena have been studied intensively in population dynamics. A problem that motivated this paper was related to the carbon content of an ecosystem — recently some authors have identified disturbances (extreme droughts, fires, insect outbreaks, etc.) as key forces driving the dynamics of carbon [1–4], and, more generally, as a factor in the dynamics of vegetation (see, e.g., [5], and the recent papers by D'Odorico et al. [6] and Beckage et al. [7]).

Another situation in which disturbances play an important role is the stochastic phenotype switching in microbial populations in response to sudden catastrophic events in the environment (see, e.g., [8,9], or, for a more mathematical exposition, [10]). Visco et al. [11] recently suggested a mathematical model of the dynamics of such systems that is somewhat reminiscent of the model we consider in this paper. We, however, will use the carbon content of an ecosystem as a motivating example throughout the paper because its nature matches more closely our assumptions in the main theorems of our paper.

The amount of carbon in the ecosystem increases due to photosynthesis, and after a long time, approaches the corresponding carrying capacity of the ecosystem. Occasionally, however, a forest fire, an extreme drought, or some other process occurring on a much shorter time scale than the normal growth of plants destroys some part of the ecosystem. We call such a fast process of decimation of the forest a *disturbance*, and consider the disturbances as instantaneous events.

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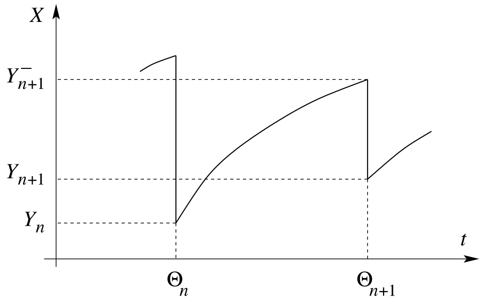

Fig. 1. On the definition of the Θ_n, Y_n^-, and Y_n (2). The process between Θ_n and Θ_{n+1} is governed by (1).

Therefore, we arrive at the following continuous-time continuous-state space stochastic process $X(t)$ modeling the carbon mass in the ecosystem. In the time between two consecutive disturbances, $X(t)$ evolves deterministically, governed by the autonomous ordinary differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}X(t) = g(X(t)). \tag{1}$$

Sometimes processes like X are called *semistochastic* in the literature. We will need that the solution of the differential equation (1) be an invertible function, so we impose the condition that the function g be strictly positive; it is allowed to be zero only at the ends of the interval where $X(t)$ is allowed to vary (as in the cases (6) and (7) below). In the context of the carbon content problem, to account for the constant growth rate of the plants and the saturation effects (due to finite carrying capacity), one can take, for example, $g(x) = 1 - x$ (cf. Eq. (6) below).

We assume that $X(t)$ is non-negative, which is the case in many applications. The quantity $X(t)$ changes with a downward jump at some discrete set of random times $\Theta_1 < \Theta_2 < \Theta_3 < \cdots$. The random times Θ_j form a renewal process. If the cause of the disturbance is natural, one can perhaps assume that Θ_j's come from a Poisson process, but one can, for example, consider the case of controlled forest fires performed at scheduled times (unless a natural fire occurs), in which case the process will not be Poisson. We assume that the fraction of the forest that is destroyed in the disturbance – termed the *severity* of the disturbance – is a continuous random variable with a known distribution supported on [0, 1). To simplify our considerations, we assume that the times of occurrence of the disturbances do not depend on the state of the system (i.e., on $X(t)$).

To formulate the questions we study in this paper, we need to introduce some notation. We define the pre-disturbance levels Y_n^- and the post-disturbance levels Y_n of the process by

$$Y_n^- := \lim_{t\uparrow\Theta_n} X(t), \qquad Y_n := \lim_{t\downarrow\Theta_n} X(t); \tag{2}$$

as represented pictorially in Fig. 1. Let the severity of the nth disturbance be determined by a continuous random variable U_n relating the pre- and post-disturbance levels:

$$Y_n = U_n Y_n^-. \tag{3}$$

Clearly, U_n should be supported on the interval [0, 1), and it is reasonable to assume that these random variables are independent and identically distributed. We also want that U_n be independent of the process $X(t)$. Of course, the practical measurements of the distributions of the inter-disturbance times and the severity of the disturbances in practical situations is a complicated issue (see, e.g., [12]).

One meaningful question that can be asked is to find the distributions of the pre- and post-disturbance levels. Another interesting problem is to find the fraction of time the process X spends in the long run in a certain measurable set $A \subset \mathbb{R}$. We will assume that the long-time distribution of X can be described by a p.d.f. f_X defined as

$$\int_A f_X(x)\,\mathrm{d}x = \lim_{T\to\infty} \frac{1}{T}\,\mu\left(\{t \in [0, T] : X(t) \in A\}\right), \tag{4}$$

for any Borel set A (where μ stands for the Lebesgue measure), whenever this limit exists.

We will call p.d.f.'s like f_X *stationary* or *invariant* distributions, and the measures they define, *invariant* measures.

In this paper, we derive an explicit expression for the conditional probability density function $f_{Y_{n+1}|Y_n}$ relating two consecutive post-disturbance levels, in terms of the p.d.f. f_T of the inter-disturbance times T_n, the p.d.f. f_U of the severities U_n, and the function g in (1). This function is the kernel in an integral equation for the stationary probability density function f_Y of the post-disturbance levels Y_n. To solve the integral equation for f_Y, we transform it to a differential equation that is easier to solve. We also derive an explicit expression for f_X (4) in terms of f_Y, f_U, and g.

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