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Non-simultaneous blow-up and blow-up rates for reaction–diffusion equations

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ABSTRACT

This paper considers blow-up solutions for reaction–diffusion equations, complemented by homogeneous Dirichlet boundary conditions. It is proved that there exist initial data such that one block or two (separated or contiguous) blocks of n components blow up simultaneously while the others remain bounded. As a corollary, a necessary and sufficient condition is obtained such that any blow-up must be the case for at least two components blowing up simultaneously. We also show some other exponent regions, where any blow-up of $k(\in \{1, 2, \dots, n\})$ components must be simultaneous. Moreover, the corresponding blow-up rates and sets are discussed. The results extend those in Liu and Li [B.C. Liu, F.J. Li, Non-simultaneous blow-up of n components for nonlinear parabolic systems, J. Math. Anal. Appl. 356 (2009) 215–231].

1. Introduction

In this paper, we consider the reaction–diffusion system

$$\begin{cases} (u_i)_t = \Delta u_i + u_i^{p_i}(0,t)u_{i+1}^{q_{i+1}}(0,t), & (x,t) \in B_R \times (0,T), \\ u_i = 0, & (x,t) \in \partial B_R \times (0,T), \\ u_i(x,0) = u_{i,0}(x), & i = 1,2,\dots,n,\ n \geq 2,\ x \in B_R, \\ u_{n+1} := u_1, \qquad p_{n+1} := p_1, \qquad q_{n+1} := q_1, \end{cases} \tag{1.1}$$

where B_R denotes a ball in $\mathbf{R}^N$ with radius R, centered at the origin. The exponents $p_i, q_i,\ i = 1, 2, \dots, n$ are all non-negative; $u_{i,0}(x), i = 1, 2, \dots, n$ are smooth and nonnegative functions, vanishing on the boundary. Let T be the maximal existence time for positive solutions of system (1.1). Proof of the existence of classical positive solutions to (1.1) can be found in [1], and also obtained by the methods developed in [2].

Nonlinear parabolic systems like (1.1) come from population dynamics, chemical reactions, heat transfer, etc., where the n components represent, for example, the densities of biological populations during migrations, concentrations of chemical reactants, and temperatures of materials during heat propagation, where the nonlinearities in dynamical systems represent uniform reactions on the whole domain dictated by the values at a single point (see [3,4]).

Recently, Li and Wang [5], and Zhao and Zheng [6], independently, considered blow-up solutions of reaction–diffusion equations of the form

$$u_t = \Delta u + u^{p_1}(x_0,t)v^{q_2}(x_0,t), \qquad v_t = \Delta v + u^{q_1}(x_0,t)v^{p_2}(x_0,t), \quad (x,t) \in \Omega \times (0,T), \tag{1.2}$$

complemented by homogeneous Dirichlet boundary conditions. The following results were obtained. If $q_2 \geq p_2 - 1 > 0$ and $q_1 \geq p_1 - 1 > 0$, then u and v blow up simultaneously. If u and v blow up simultaneously, then the exponents satisfy

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$q_2 \geq p_2 - 1$ and $q_1 \geq p_1 - 1$, or $q_2 < p_2 - 1$ and $q_1 < p_1 - 1$. The uniform blow-up profiles are obtained with precise coefficients for $q_2 \geq p_2 - 1$ and $q_1 \geq p_1 - 1$, or $q_2 < p_2 - 1$ and $q_1 < p_1 - 1$. Moreover, in the regions $q_2 > p_2 - 1$, $q_1 > p_1 - 1$, and $p_1 q_2 > (1 - p_1)(1 - p_2)$, or $q_2 < p_2 - 1$ and $q_1 < p_1 - 1$, the boundary layer sizes for blow-up solutions are discussed. For other related works, one can refer to [7–15]. For surveys, one can refer to [16,17].

Non-simultaneous blow-up phenomena have been observed by Quirós and Rossi [18] for equations of the form

$$u_t = \Delta u + u^{p_1} v^{q_2}, \qquad v_t = \Delta v + u^{q_1} v^{p_2}, \quad (x, t) \in \mathbb{R}^N \times (0, T). \tag{1.3}$$

They proved that there exist initial data such that u blows up alone if $p_1 > q_1 + 1$. If u blows up at some point $x' \in \mathbb{R}^N$ while v remains bounded and $u(x, t) \geq c(T - t)^{-\frac{1}{p_1 - 1}}$ for $|x - x'| \leq K\sqrt{T - t}$, then $p_1 > q_1 + 1$. The blow-up rates for (1.3) were obtained by Guo, Sasayama, and Wang in [19]. For other studies on non-simultaneous blow-up, one can refer to [20–24], amongst others.

In [25], Wang discussed n-componential parabolic equations

$$(u_i)_t = \Delta u_i + u_{i+1}^{q_{i+1}}, \; i = 1, 2, \ldots, n, \quad (x, t) \in \Omega \times (0, T), \tag{1.4}$$

together with null Dirichlet conditions, where Ω is a general bounded domain of $\mathbf{R}^N$; $u_{n+1} := u_1, p_{n+1} := p_1, q_{n+1} := q_1$. If $\prod_{i=1}^n q_i > 1$ and $(u_i)_t \geq 0$, then there exist constants $C, c > 0$ such that

$$c(T - t)^{-\lambda_i} \leq \max_{\Omega} u_i(\cdot, t) \leq C(T - t)^{-\lambda_i}, \quad i = 1, 2, \ldots, n, \tag{1.5}$$

with $\lambda_i = \frac{1 + q_i + \sum_{l=i+1}^{n+i-2} q_i \cdots q_l}{\prod_{i=1}^n q_i - 1} > 0$. Eq. (1.4) with $\Omega = \mathbf{R}^N$ was discussed by Fila and Quittner [26]. They obtained that there exists some positive constant C such that $u_i(x, t) \leq C(T - t)^{-\lambda_i}$, $i = 1, 2, \ldots, n$, provided that $\max\{\lambda_1, \lambda_2, \ldots, \lambda_n\} > N/2$. Pedersen and Lin [27] discussed n-componential parabolic equations of the form

$$(u_i)_t = \Delta u_i + u_{i+1}^{q_{i+1}}(x_0, t), \quad i = 1, 2, \ldots, n, \; (x, t) \in \Omega \times (0, T), \tag{1.6}$$

subject to null Dirichlet conditions, where x_0 is a point in Ω; $q_i \geq 1, \; i = 1, 2, \ldots, n$; $u_{n+1} := u_1, p_{n+1} := p_1, q_{n+1} := q_1$. The blow-up rate (1.5) was obtained. Moreover, boundary layer estimates were considered.

The authors of the present paper discussed heat equations in [24], coupled via a nonlinear Neumann boundary flux

$$\frac{\partial u_i}{\partial \eta} = u_i^{p_i} u_{i+1}^{q_{i+1}}, \quad i = 1, 2, \ldots, n, \; (x, t) \in \partial B_R \times (0, T),$$

where $u_{n+1} := u_1, p_{n+1} := p_1, q_{n+1} := q_1$. Non-simultaneous blow-up means that at least one of the n components remains bounded when blow-up occurs at some finite time. Due to different blow-up mechanisms among the n components, the solution presents quite different non-simultaneous blow-up phenomena. It is interesting that, even for the same blow-up components, the blow-up rates would also be different. Three main results are obtained:

- The existence of only one of the n components blowing up while the others remain bounded.
- The existence of only two components blowing up simultaneously.
- The occurrence of non-simultaneous or simultaneous blow-up for all initial data in some exponent regions.

As well as facing the difficulties of classification of the exponents, another difficulty needs to be overcome: the choice of suitable initial data quantitatively in order to keep the blow-up time, satisfying both the monotone and compatibility conditions. In the present paper, we obtain the following results, which extend the results in [24].

- There exist suitable initial data such that one block or two (separated or contiguous) blocks of the n components blow up simultaneously.
- Any blow-up must be the case for $k + 1 (\in \{1, 2, \ldots, n - 1\})$ components blowing up while the other $(n - k - 1)$ components remain bounded.
- If all $p_i \leq 1$ and $\prod_{j=1}^n q_j > \prod_{j=1}^n (1 - p_j)$, then any blow-up must be simultaneous.

In addition, the blow-up rates and blow-up set are discussed.

2. Main results

Because of the nonlinear $u_i^{p_i}(0, t)$, non-simultaneous blow-up may occur for system (1.1). In fact, u_i may blow up at some finite time by itself if the exponent $p_i > 1$. Considering blow-up criteria to (1.4) and (1.6), one can find that, if all $p_i \leq 1$ and $\prod_{j=1}^n q_j > \prod_{j=1}^n (1 - p_j)$, the solutions of (1.1) also can blow up.