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A note on phase synchronization in coupled chaotic fractional order systems

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ABSTRACT

The dynamic behaviors of fractional order systems have received increasing attention in recent years. This paper addresses the reliable phase synchronization problem between two coupled chaotic fractional order systems. An active nonlinear feedback control scheme is constructed to achieve phase synchronization between two coupled chaotic fractional order systems. We investigated the necessary conditions for fractional order Lorenz, Lü and Rössler systems to exhibit chaotic attractor similar to their integer order counterpart. Then, based on the stability results of fractional order systems, sufficient conditions for phase synchronization of the fractional models of Lorenz, Lü and Rössler systems are derived. The synchronization scheme that is simple and global enables synchronization of fractional order chaotic systems to be achieved without the computation of the conditional Lyapunov exponents. Numerical simulations are performed to assess the performance of the presented analysis.

1. Introduction

Chaos has been investigated and studied in mathematical and physical communities in the last few decades because of its useful applications in many fields such as secure communication, data encryption, flow dynamics and biomedical engineering [1]. Research efforts have been devoted to chaos control and chaos synchronization problems in nonlinear science because of its extensive applications [2–5]. Different types of synchronization problems have been observed and studied in various chaotic systems, such as identical synchronization, generalized synchronization, phase synchronization and so on [6–11]. Complete synchronization of identical systems occurs when the states of coupled systems coincide. Generalized synchronization implies that the output of one system is associated with a given function of the output of another system. Phase synchronization occurs when the synchronized states of coupled systems are not identical, the amplitudes of oscillations often remain chaotic and practically uncorrelated, and only their phases evolve in synchrony. Thus, it is mostly close to synchronization of periodic oscillators.

On the other hand, the development of models based on fractional order differential systems has recently gained popularity in the investigation of dynamical systems. Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. The main reason for using integer-order models was the absence of solution methods for fractional differential equations. The advantages or the real objects of the fractional order systems [12] are that we have more degrees of freedom in the model and that a "memory" is included in the model.

Recently, studying fractional order systems has become an active research area. The chaotic dynamics of fractional order systems began to attract much attention in recent years. It has been shown that the fractional order systems, as

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generalizations of many well-known systems, can also behave chaotically, such as the fractional Duffing system [13], the fractional Chua system [12,14], the fractional Rössler system [15], the fractional Chen system [16,17], the fractional Lorenz system [18], fractional Arneodo's system [19] and the fractional Lü system [20]. In [14–16] it has been shown that some fractional order systems can produce chaotic attractors with order less than 3.

Moreover, recent studies show that chaotic fractional order systems can also be synchronized [21–32]. In many literatures, synchronization among fractional order systems is only investigated through numerical simulations that are based on the stability criteria of linear fractional order systems, such as the work presented in [24–26], or based on Laplace transform theory, such as the work presented in [27–29]. Very recently some researchers have shown the existence of phase synchronization in chaotic fractional differential equations [33,34]. The main aim of this paper is to study the phase synchronization of coupled chaotic Caputo based fractional order systems. We have employed the stability results of linear fractional order systems in our analysis to achieve phase and complete synchronization. An active control scheme, which consists of a nonlinear dynamic feedback controller, has been proposed. The effectiveness of the proposed scheme is demonstrated via its application to the phase synchronization of the fractional models of Lorenz, Lü and Rössler systems.

1.1. Basic concepts

There are several definitions of a fractional derivative of order $\alpha > 0$ [35–39]. The two most commonly used are the Riemann–Liouville and Caputo definitions. Each definition uses Riemann–Liouville fractional integration and derivatives of whole order. The difference between the two definitions is in the order of evaluation. The Riemann–Liouville fractional integral operator of order $\alpha \geq 0$ of the function $f(t)$ is defined as,

$$J^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) \mathrm{d}\tau, \quad \alpha > 0, \ t > 0. \tag{1}$$

Some properties of the operator J^{α} can be found, for example, in [36,38]. We recall only the following, for $\mu \geq -1$, $\alpha, \beta \geq 0$ and $\gamma > -1$, we have,

$$J^{\alpha} J^{\beta} f(t) = J^{\alpha+\beta} f(t),$$
$$J^{\alpha} t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}.$$

In this study, Caputo definition is used and the fractional derivative of $f(t)$ is defined as,

$$D^{\alpha} f(t) = J^{m-\alpha} D^m f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha-m+1}} \mathrm{d}\tau, \tag{2}$$

for $m - 1 < \alpha \leq m, \ m \in \mathbb{N}, \ t > 0$.
Caputo's definition has the advantage of dealing properly with initial value problems in which the initial conditions are given in terms of the field variables with their integer order which is the case in most physical processes.

1.2. Stability analysis of fractional systems

Stability analysis of fractional order systems, which is of great interest in control theory, has been thoroughly investigated where necessary and sufficient conditions have been derived [40–43] (see also references therein). In this section, we recall the main stability results. For this object, we consider the following n dimensional fractional order system,

$$\begin{cases} \dfrac{\mathrm{d}^{\alpha_1} x_1}{\mathrm{d}t^{\alpha_1}} = f_1(x_1, x_2, \ldots, x_n), \\ \dfrac{\mathrm{d}^{\alpha_2} x_2}{\mathrm{d}t^{\alpha_2}} = f_2(x_1, x_2, \ldots, x_n), \\ \vdots \\ \dfrac{\mathrm{d}^{\alpha_n} x_n}{\mathrm{d}t^{\alpha_n}} = f_n(x_1, x_2, \ldots, x_n), \end{cases} \tag{3}$$

where α_i is a rational number between 0 and 1 and $\frac{\mathrm{d}^{\alpha_i}}{\mathrm{d}t^{\alpha_i}}$ is the Caputo fractional derivative of order α_i, for $i = 1, 2, \ldots, n$. Assume that $\alpha_i = l_i/m_i$, $(l_i, m_i) = 1$, $l_i, m_i \in \mathbb{N}$, for $i = 1, 2, \ldots, n$. Let m be the least common multiple of the denominators m_i's of α_i's.

First, if the system (3) is a linear system, that is $[f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_n(\mathbf{x})]^T = [a_{ij}]_{i,j=1}^n \mathbf{x} = A\mathbf{x}$, where $\mathbf{x} \in \mathbb{R}^n$, then we have the following results:

- If $\alpha = \alpha_1 = \alpha_2 = \cdots = \alpha_n$, then the fractional order system (3) is asymptotically stable iff $|\arg(\mathrm{spec}(A))| > \alpha\pi/2$. In this case the components of the state decay towards 0 like $t^{-\alpha}$ [40].
- If α_i's are rational numbers between 0 and 1, then the system (3) is asymptotically stable if all roots λ of the equation $\det\big(\mathrm{diag}(\lambda^{m\alpha_1}, \lambda^{m\alpha_2}, \ldots, \lambda^{m\alpha_n}) - A\big) = 0$ satisfy $|\arg(\lambda)| > \gamma\pi/2$, where $\gamma = 1/m$ [41].