Nonlinear Analysis: Real World Applications 13 (2012) 860–881

Contents lists available at SciVerse ScienceDirect

Nonlinear Analysis: Real World Applications

journal homepage: www.elsevier.com/locate/nonrwa

Random fuzzy differential equations under generalized Lipschitz condition

Marek T. Malinowski

Faculty of Mathematics, Computer Science and Econometrics, University of Zielona Góra, Szafrana 4a, 65-516 Zielona Góra, Poland

ARTICLE INFO

Article history:
Received 27 January 2011
Accepted 21 August 2011

Keywords:
Random fuzzy differential equation
Fuzzy differential equation
Fuzzy derivative
Fuzzy stochastic process
Fuzzy random variable
Fuzziness and randomness in modelling

ABSTRACT

We present the studies on two kinds of solutions to random fuzzy differential equations (RFDEs). The different types of solutions to RFDEs are generated by the usage of two different concepts of fuzzy derivative in the formulation of a differential problem. Under generalized Lipschitz condition, the existence and uniqueness of both kinds of solutions to RFDEs are obtained. We show that solutions (of the same kind) are close to each other in the case when the data of the equation did not differ much. By an example, we present an application of each type of solutions in a population growth model which is subjected to two kinds of uncertainties: fuzziness and randomness.

1. Introduction

Knowledge about differential equations is often incomplete or vague. For example, initial conditions or the values of functional relationships may not be known precisely. In such a situation, the usage of fuzzy differential equations (FDEs) is a natural way to model dynamical systems under possibilistic uncertainty. FDEs is a very important topic from the theoretical point of view (see e.g. [1,2] and references therein) as well as of their applications, for example, in civil engineering [3], in modelling hydraulic [4], in population models [5,6], in modelling of a three-phase induction motor [7]. FDEs were first formulated by Kaleva [8,9]. He used the concept of H-differentiability which was introduced by Puri and Ralescu [10], and obtained the existence and uniqueness theorem for a solution of FDE under the Lipschitz condition, whereas in [9] he characterized those subsets of the fuzzy set space in which the Peano theorem is valid. Since then there appeared a lot of papers concerning different approaches to the theory of FDEs as well as applications of FDEs, fuzzy dynamics, fuzzy differential inclusions and numerical methods for FDEs (see e.g. [11–46]). A rich collection of results from the theory of FDEs is contained in the monographs of Lakshmikantham and Mohapatra [1], and Diamond and Kloeden [2].

In this paper, we will consider random fuzzy differential equations (RFDEs) as they can provide good models of dynamics of real phenomena which are subjected to two kinds of uncertainties: randomness and fuzziness, simultaneously. The first source of uncertainty is connected with the uncertainty in prediction of the outcome of an experiment. Randomness intends to break the law of causality and the probabilistic methods are applied in its analysis. Second type of uncertainty – fuzziness – means nonstatistical inexactness that is due to subjectivity and imprecision of human knowledge rather than to the occurrence of random events. It is caused by the lack of sharply defined criteria of membership in the sets of some considered space (a simple example of a fuzzy set could be a class of real numbers which are much greater than 1). Fuzziness intends to break the law of excluded middle and is appropriately treated by the fuzzy set theory. The probability and fuzzy set theories team up in the concept of fuzzy random variable. This notion is a fundamental one in the analysis of RFDEs.

In the literature, one can find various definitions of fuzzy random variables. For the first time the concept of fuzzy random variable was proposed by Kwakernaak [47]. Further, it was used by Kruse and Meyer [48]. In [49–51,2], there appear two

E-mail address: m.malinowski@wmie.uz.zgora.pl.

1468-1218/$ – see front matter
doi:10.1016/j.nonrwa.2011.08.022

notions of measurability of fuzzy mappings. The relations between different concepts of measurability for fuzzy random variables are contained in the papers of Colubi et al. [52], Terán Agraz [53], López-Díaz and Ralescu [54]. In this paper, we will use a definition of fuzzy random variable which was introduced by Puri and Ralescu [51]. This definition is currently the most often used in probabilistic and statistical aspects of the theory of fuzzy random variables.

The authors of [55–58] considered the differential equations where the two kinds of uncertainties (randomness and fuzziness) were incorporated. Feng [55] considered fuzzy stochastic differential systems using a notion of mean-square derivative (which is different from our derivative) and mean-square integral of second-order fuzzy stochastic processes introduced by himself in [59]. In his setting, the fuzzy random variable comes from a narrower class than ours. Using the Banach fixed point theorem, the existence and uniqueness of the Cauchy problem is obtained under an assumption that the coefficients satisfy a condition with the Lipschitz constant. In [56], the existence and uniqueness of the solution for RFDE's with non-Lipschitz coefficients is proved. The values of fuzzy mappings are in the space of fuzzy sets of a reflexive separable Banach space. However, only the autonomous case is treated, where right-hand side is non-random. Malinowski [57] considered RFDEs with fuzzy derivative defined as in [10]. The coefficients of the equation were random fuzzy functions, also the initial condition was treated as a fuzzy random variable. With an assumption that the right-hand side of the equation satisfies a global Lipschitz-type condition the existence and uniqueness of the solution to RFDEs was proved. In [58], Malinowski examined RFDEs with two kinds of fuzzy derivative. Supposing that the Lipschitz condition holds on bounded sets, the existence and uniqueness of a local solution to RFDEs was obtained. Under an assumption that the right-hand side of the equation satisfies some integrability condition, the existence of at least one local solution was proved. This result was then applied in a demonstration of the existence of at least one global solution to RFDEs.

In this paper, we study two kinds of solutions to RFDEs. The different types of solutions to RFDEs are generated by the usage of two different concepts of fuzzy derivative. This direction of research is motivated by the results of Bede and Gal [17], Chalco-Cano and Román-Flores [21] concerning deterministic FDEs with generalized fuzzy derivative and recently by the paper of Stefanini and Bede [60] where two types of solutions to interval differential equations were investigated. Under generalized Lipschitz condition we obtain the existence and uniqueness of the solutions to both kinds of RFDEs. To prove this assertion we use an idea of successive approximations which has been applied in [17] for FDEs and in [57,58] for RFDEs. Then we study a qualitative behaviour of the solutions. Finally, by an example of application, we show a practical need of considerations of the two types of solutions to RFDEs.

The paper is organized as follows. In Section 2, we collect the fundamental notions and facts about fuzzy set space, fuzzy differentiation and integration. We recall the notions of fuzzy random variable and fuzzy stochastic process. In Section 3, we discuss the RFDEs with a first type of fuzzy derivative, whereas in Section 4, the RFDEs with a second type of fuzzy derivative are investigated. For both cases, under suitable conditions we prove the existence and uniqueness of the solutions to RFDEs. We do an analysis of the behaviour of the solutions when data of the equation are subjected to errors. In Section 5, we apply two kinds of solutions to RFDEs in a population model with uncertainties. Finally, Section 6 concludes the paper.

2. Preliminaries

In this section, we give some notations and properties related to fuzzy set space, and summarize the major results for integration and differentiation of fuzzy-set-valued mappings. We recall also the notions of fuzzy random variable and fuzzy stochastic process.

Let $\mathcal{K}(\mathbb{R}^d)$ denote the family of all nonempty compact convex subsets of $\mathbb{R}^d$ and define addition and scalar multiplication in $\mathcal{K}(\mathbb{R}^d)$ as usual, i.e. for $A, B \in \mathcal{K}(\mathbb{R}^d)$ and $\lambda \in \mathbb{R}$

$$A + B := \{a + b \mid a \in A, b \in B\}, \qquad \lambda A := \{\lambda a \mid a \in A\}.$$

The Hausdorff metric d_H in $\mathcal{K}(\mathbb{R}^d)$ is defined as follows

$$d_H(A, B) := \max\left\{\sup_{x\in A}\inf_{y\in B}\|x-y\|, \sup_{y\in B}\inf_{x\in A}\|x-y\|\right\}, \quad A, B \in \mathcal{K}(\mathbb{R}^d),$$

where $\|\cdot\|$ denotes usual Euclidean norm in $\mathbb{R}^d$. The metric space $(\mathcal{K}(\mathbb{R}^d), d_H)$ is complete, separable and locally compact (cf. [1,61]).

A fuzzy set u in $\mathbb{R}^d$ is characterized by its membership function (denoted by u again) $u: \mathbb{R}^d \to [0, 1]$ and $u(x)$ (for each $x \in \mathbb{R}^d$) is interpreted as the degree of membership of element x in the fuzzy set u. As the value $u(x)$ expresses a "degree of membership of x in" or a "degree of satisfying by x a property", one can work with imprecise information.

Denote

$$E^d := \{u: \mathbb{R}^d \to [0, 1] \mid u \text{ satisfies (i)–(iv) below}\},$$

(i) u is normal, i.e. there exists $x_0 \in \mathbb{R}^d$ such that $u(x_0) = 1$,
(ii) u is fuzzy convex, i.e. $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}$ for any $x, y \in \mathbb{R}^d$ and $\lambda \in [0, 1]$,
(iii) u is upper semicontinuous,
(iv) $[u]^0 := \mathrm{cl}\{x \in \mathbb{R}^d \mid u(x) > 0\}$ is compact, where cl denotes the closure in $(\mathbb{R}^d, \|\cdot\|)$.

Although elements of E^d are often called the fuzzy numbers, we shall just call them the fuzzy sets.