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A directed weighted complex network for characterizing chaotic dynamics from time series

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ABSTRACT

We propose a reliable method for constructing a directed weighted complex network (DWCN) from a time series. Through investigating the DWCN for various time series, we find that time series with different dynamics exhibit distinct topological properties. We indicate this topological distinction results from the hierarchy of unstable periodic orbits embedded in the chaotic attractor. Furthermore, we associate different aspects of dynamics with the topological indices of the DWCN, and illustrate how the DWCN can be exploited to detect unstable periodic orbits of different periods. Examples using time series from classical chaotic systems are provided to demonstrate the effectiveness of our approach.

1. Introduction

Quite recently, the approach of complex networks has been incorporated into the study of a dynamical process underlying a given experimental time series of time dependent complex systems [1–17], motivated by the fact that complex networks are capable of modeling and characterizing many types of systems in nature and technology that contain a large number of components interacting with each other in a complicated manner [18–32]. Such an approach has already been implemented successfully, for instance, for data from climate system [33], multi-phase flow [34,35], human gait [36] and human brain system [37]. Despite the existing results, significant challenges in applying complex network theory to the investigation of complex experimental time series remain. For example, characterizing and detecting the unstable periodic orbits (UPOs) in terms of a complex network approach is still a challenging problem of continuing interest.

In this paper, we propose a unique *directed weighted complex network* (DWCN) for characterizing complicated dynamics from time series motivated by the consideration that different nodes and links may play distinct functional roles in a complex network. Specifically, given a set of time series from various dynamic systems, our first step is to construct a directed weighted complex network using a heuristic theory for determining the threshold. We then associate different aspects of chaotic dynamics with the topological indices of the DWCN and demonstrate that the DWCN can be exploited to detect unstable periodic orbits of low periods. Examples using time series from classical chaotic systems are provided to demonstrate the effectiveness of our approach.

2. Construction of a directed weighted complex network

We start from construction of the DWCN. Our first step is phase space reconstruction. Given a time series $z(it)$ ($i = 1, 2, \ldots, M$), where t is the sampling interval and M is the sample size, we construct a sequence of phase space vectors

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according to the standard delay-coordinate embedding method [38–40]:

$$\begin{aligned}\overrightarrow{X_k} &= \{x_k(1), x_k(2), \ldots, x_k(m)\} \\ &= \{z(kt), z(kt+\tau), \ldots, z(kt+(m-1)\tau)\}\end{aligned} \tag{1}$$

where τ is the delay time, m is the embedding dimension, $k = 1, 2, \ldots, N$, and $N = M - (m-1)\tau/t$ is the total number of vector points in the reconstructed phase space. To construct a network, we then regard each vector point as a node and use the phase space distance to determine the edges. Given two vector points $\overrightarrow{X_i}$ and $\overrightarrow{X_j}$ $(i > j)$, the phase space distance is defined to be

$$d_{ij} = \sum_{n=1}^{m} \left\| X_i(n) - X_j(n) \right\| \tag{2}$$

where $X_i(n) = z(i*t + (n-1)\tau)$ is the nth element of $\overrightarrow{X_i}$. This generates, for all nodes (vector points) in the network, a distance matrix $\mathbf{D} = (d_{ij})$, where $i > j$. By choosing a critical threshold value r_c, we obtain the connections of the network: an edge connecting node i and j $(i > j)$ exists if $|d_{ij}| \le r_c$; while there is no edge between i and j if $|d_{ij}| > r_c$. We regard the time $(i-j)*t$ as the weight of an edge that connected nodes i and j and the edge direction is from node i to j. Finally, we obtain the weight matrix $\mathbf{W} = (w_{ij})$, where $w_{ij} = 0$ means node i and j are not connected; otherwise, $w_{ij} \ne 0$ implies that an edge from node i to j exists and the edge weight is $w_{ij} = (i-j)*t$. The topology of the reconstructed DWCN is determined entirely by $\mathbf{W}$.

A key issue in extracting DWCN from time series is then the choice of the critical threshold r_c. In this paper, we exploit the normalized maximum size of subgraph to determine the critical threshold. Take the Tent map

$$f(x) = \begin{cases} 2x, & \text{if } x < 1/2 \\ 2(1-x), & \text{if } x \ge 1/2 \end{cases} \tag{3}$$

as an example; here, we theoretically demonstrate how to properly select the critical threshold. The locations of the periodic orbits of the Tent map can be obtained explicitly. At each iteration, the map has two line segments of slope 2 and −2 in the unit square of the plane x_{n+1} versus x_n. The pth-iterated map has 2^p line segments in the unit square. Fixed points of the pth-iterated map, which contain all periodic orbits of period p, are located at cross points of these 2^p line segments with the line $x_{n+1} = x_n$. Thus, we have the following set of points that belong to different periodic orbits of period p:

$$X_p(j) = \begin{cases} 2j/(2^P + 1), & \text{if } j = 1, 2, 3, \ldots, 2^{P-1} \\ 2j/(2^P - 1), & \text{if } j = 1, 2, 3, \ldots, 2^{P-1} - 1. \end{cases} \tag{4}$$

Starting with one such point, one can obtain the remaining $p-1$ points on the orbit by iterating the Tent map. We obtain all the orbit points from period 1–11 considering the fact that the chaotic dynamics mainly focus on the unstable periodic orbits of low periods. Then we calculate the minimum distance from the point i on the orbit q_l of period P_l to the points of other orbits

$$L_{P_l}^{q_l}(i) = \min\left(\left\{ \left[X_{P_l}^{q_l}(i) - X_{P_k}^{q_k}(j) \right]^2 + \left[f(X_{P_l}^{q_l}(i)) - f(X_{P_k}^{q_k}(j)) \right]^2 \right\}^{1/2} \middle| \begin{array}{l} P_k = 1, 2, \ldots, 11 \\ q_k = 1, 2, \ldots, 2^{P_k} - 1 \\ (P_k, q_k) \ne (P_l, q_l) \end{array} \right) \tag{5}$$

where $2^{p_k-1} - 1$ is the number of orbit of period P_k. $L_{P_l}^{q_l}(i)$ is the minimum distance for point i. Then we calculate the minimum distance for all points on the orbits of different periods and select the maximum value of the obtained minimum distances

$$r_c = \max\left(L_{p_l}^{q_l}(i) \middle| P_l = 1, 2, \ldots, 11 \; q_l = 1, 2, \ldots, 2^{P_l} - 1 \; i = 1, 2, \ldots, P_l \right) = 0.00155 \tag{6}$$

as the network critical threshold in that it is the smallest one that can guarantee the skeleton of the network. So, we have theoretically calculated the critical threshold from the equations for Tent map system.

To cast light into the time series of length 8000 from the Tent map, we now carry out the *normalized maximum size of subgraph* (NMSS) distribution to determine the critical threshold, as follows. We reconstruct the network and study the distributions of NMSS with respect to the variation in threshold. For a network from time series, the increase of NMSS will reach the maximum rate when the threshold approaches the smallest value that can preserve the network skeleton. By analysis, we find that when r_c approaches 0.00175, the NMSS reaches maximum increase rate. Thus, the $r_c = 0.00175$ is the critical threshold for the Tent map and the result is in agreement with that obtained from theoretical calculation $r_c = 0.00155$. Furthermore, we use the NMSS distribution study the 2x mod 1 map and the results indicate that, consistently with the one theoretically calculated from equations $r_c = 0.0021$, the critical threshold obtained from NMSS distribution is $r_c = 0.00205$, confirming that the NMSS distribution could be a faithful approach for determining the network critical threshold.