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Variational level set methods for image segmentation based on both L^2 and Sobolev gradients

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ABSTRACT

Variational level set methods for image segmentation involve minimizing an energy functional over a space of level set functions using a continuous gradient descent method. The functional includes the internal energy (curve length, usually) for regularization and the external energy that aligns the curves with object boundaries. Current practice is, in general, to minimize the energy functional by calculating the L^2 gradient of the total energy. However, the Sobolev gradient is particularly effective for minimizing the curve length functional by the gradient descent method in that it produces the solution in a single iteration. In this paper, we thus propose to use the Sobolev gradient for the internal energy (curve length), while still using the L^2 gradient for the external energy. The test results show that the "L^2 plus Sobolev" gradient scheme is significantly more computationally efficient than the methods only based on the L^2 gradient.

1. Introduction

Image segmentation is one of the fundamental and important tasks in image analysis and computer vision. Given an image, the segmentation goal is to separate the image domain into dissimilar regions, each of which has a consistent trait (intensity, color or texture, etc.) that is different from other regions in the image. Once a decision is made on the desired trait, various methods are available to reach the segmentation goal. This paper will focus on variational level set methods [1].

The variational level set methods [1–13] have been well established and widely used in various image applications (especially in the medical imaging field [9,14]) since Chan and Vese [2] primarily presented the "active contour without edges" model. The basic idea behind these methods is explained as follows. In the level set method [15], the contours are first implicitly represented by the zero level set of a higher dimensional function, usually referred to as the level set function. The segmentation problem is then formulated in terms of minimizing (or at least finding critical points of) an energy functional which typically includes the internal energy that smoothes the level set function and the external energy that aligns the zero level set with object boundaries. The level set functions evolve according to an evolution partial differential equation (PDE), which is derived from the minimization of the energy functional by calculating the L^2 (ordinary) gradient of energy functional and using the continuous gradient descent method (see [1–10,12,13] for example).

Many variational level set methods usually need a large number of iterations for level set functions to come to a steady state. There is no simple answer that applies generally to date [8]. One important reason for such slow convergence may be the L^2 gradient itself because the L^2 gradient descent generally converges slowly when implemented numerically [16]. However, the gradient descent method itself seems to be the method of choice, at least for the segmentation problem [11]. Fortunately, with the Sobolev gradient rather than the L^2 gradient, the gradient descent method is often very effective [17]. Richardson [16] has discussed the application of the Sobolev gradient method to a variety of partial differential equations

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that arise in image processing; his test results demonstrate substantial benefits in computational efficiency. Renka [11] has also demonstrated this computational efficiency with test results for the Sobolev gradient descent implementation of the level set evolution without reinitialization [3], a well known variational level set method for image segmentation.

Following [11], but in different ways, we discuss the application of the Sobolev gradient to some variational level set methods, in which the internal energy is defined by the length of contours (zero level set) (e.g., [2,4–7,9,10,12]). The Sobolev gradient is particularly effective for the minimum-length curve problem in that the curve corresponding to a single step of steepest Sobolev descent with step length 1 is exactly the solution to the problem [18–21]. The surprising observation is the starting point for our work in this paper. In this study, we propose to use the Sobolev gradient for the internal energy (curve length), while still using L^2 gradient for the external energy. The test results for an implementation of the Chan–Vese model [2] show that this "L^2 plus Sobolev" scheme is significantly more efficient than the methods only based on L^2 gradient.

Variational level set methods together with a primary and prototypical example are simply reviewed in Section 2, the proposed "L^2 plus Sobolev" scheme is discussed in Section 3, and numerical methods and test results are presented in Section 4.

2. Variational level set methods

Variational level set methods for image segmentation share the common feature that the optimal segmentation is given by a minimizer or at least critical point of an energy functional defined over a space of level set functions, which generally depends on the image data and the desired traits that are used to identify the different segmented regions. The evolution PDEs of level set functions are derived from the energy functional by calculating the L^2 gradient of the energy functional and using continuous gradient descent method (see [1–10,12,13] for example).

In this section, we review mainly the Chan and Vese's "active contour without edges" model [2], a primary and prototypical example of variational level set methods [1]; it has been used in our experiments. The Chan–Vese model seeks the desired segmentation as the best piecewise constant approximation to a given image, which can be interpreted as a level set implementation of the two phase piecewise constant case of the Mumford–Shah segmentation model [22]. After the Chan–Vese model, variational level set methods for image segmentation have become quite popular [1,3–13] and widely adopted in various image applications [9,14].

In the level set method [15], a closed curve C is represented by the zero level set of a Lipschitz function $\phi(x, y)$ defined on the image domain, with the following properties:

$$\begin{cases} \phi(x,y) > 0, & (x,y) \in \text{in}\,(C) \\ \phi(x,y) = 0, & (x,y) \in C \\ \phi(x,y) < 0, & (x,y) \in \text{out}\,(C) \end{cases}$$

where in(C) and out(C) stand for the "inside" and "outside" regions divided by the curve C, respectively.

For a given image $I : \Omega \to \mathbf{R}$ where $\Omega \subset \mathbf{R}^2$ is the image domain, Chan and Vese [2] proposed to minimize the following energy functional for two-phase image segmentation:

$$\begin{aligned} \varepsilon(c_1, c_2, \phi) &= \mu L(\phi) + \varepsilon_{\text{out}}(c_1, c_2, \phi) \\ &= \mu \int_\Omega \delta(\phi)\,|\nabla\phi|\,dxdy + \left(\int_\Omega |I - c_1|^2 H(\phi) dxdy + \int_\Omega |I - c_2|^2 (1 - H(\phi))\,dxdy \right). \end{aligned} \tag{2.1}$$

Here $\mu \geq 0$ is a fixed parameter, $\phi : \Omega \to \mathbf{R}$ is the unknown level set function whose zero level set $\{(x, y) : \phi(x, y) = 0\}$ represents the unknown curve C, $L(\phi)$ denotes the length of the curve C, and $H(\phi)$ and $\delta(\phi)$ are the one-dimensional Heaviside and Dirac function, respectively.

In (2.1), the term $L(\phi)$ is the internal energy which controls the smoothness of the contour C, and the term $\varepsilon_{\text{out}}(c_1, c_2, \phi)$ is the external energy which is responsible for attracting the contour C toward the objects of interest in the image I.

Minimizing the functional in (2.1) with respect to ϕ by calculating the L^2 gradient and using continuous gradient descent method, we obtain the evolution PDE:

$$\frac{\partial\phi}{\partial t} = \delta(\phi) \left(\mu \text{div} \left(\frac{\nabla\phi}{|\nabla\phi|} \right) - (I - c_1)^2 + (I - c_2)^2 \right) \tag{2.2}$$

with the following initial and boundary conditions:

$$\phi(0, x, y) = \phi_0(x, y), \qquad \frac{\delta(\phi)}{|\nabla\phi|} \frac{\partial\phi}{\partial n} = 0 \quad \text{on } \partial\Omega.$$

Here, n denotes the exterior normal to the boundary $\partial\Omega$, $\partial\phi/\partial n$ denotes the normal derivative of ϕ at the boundary, and $\phi_0(x, y)$ is the signed distance function to the initial curve.