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Multipoint boundary value problems of Neumann type for functional differential equations

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ABSTRACT

We obtain the expression of the explicit solution to a class of multipoint boundary value problems of Neumann type for linear ordinary differential equations and apply these results to study sufficient conditions for the existence of solution to linear functional differential equations with multipoint boundary conditions, considering the particular cases of equations with delay and integro-differential equations.

1. Introduction

The importance of the study of multipoint boundary value problems for differential equations lies on their applicability to determine the behavior of many real processes in scientific and technological context. Several examples and applications of multipoint boundary value problems and nonlocal boundary value problems can be found in [1–17].

Multipoint BVPs arise, for example, in the study of generalized models for a thermostat. For instance, in [15], a second-order nonlinear differential equation with nonlocal boundary conditions whose solutions are stationary solutions for a unidimensional heat equation, modeling the temperature on a heated bar, with a controller at the right end of the bar which adds or eliminates heat depending on the temperature detected by sensors at intermediate points is studied. We also mention [18], where the study of nonlocal boundary value problems for second-order equations is shown to be effective to the modeling of a thermostat with sensors expressed as linear functionals, and the properties of Green's function are used to prove the existence of multiple positive solutions as well as nonexistence results. On the other hand, the nonlocal boundary conditions considered in [8] appear not only in the study of the stationary states of a heated bar with a thermostat (ranging from natural heat loss or fixed temperature at the left end to multipoint problems), but also in the description of the membrane response of a spherical cap (see [8] and the references therein). In [19], the author considers the problem of the existence of positive and bounded solutions to a general nonlinear boundary value problem for a class of second-order differential equations given by a function which may admit a singularity, studying its applications to the behavior of an elastic membrane.

Ref. [11] presents some results on the existence of solution to a second-order differential equation defined on $[0, 1]$ subject to the nonlocal conditions $u'(0) + au(\xi_1) = 0$, $bu'(1) + u(\xi_2) = 0$, with $0 < \xi_1 \le \xi_2 < 1$, $a,\ b \in (0, 1)$, interesting to study steady states of a heated bar, in this case, with unit length. The effect produced by the boundary conditions is the addition or elimination of heat at the endpoints, depending on the temperature measured at the instants ξ_1 and ξ_2, with the notation above. Some other heat-flow problems can be found in [5,7,8,15,20]. In [7], a nonlinear scalar heat equation

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(representing the stationary state of a model for a heated bar) is studied, and the nonlocal boundary conditions considered $u'(0) = 0$, $\beta u'(1) + u(\eta) = 0$ mean that the bar is protected from loosing or gaining heat at the initial instant $t = 0$ but, at the instant $t = 1$, dissipation occurs by the action of a controller which takes into account the measurement of temperature at the instant η. A particular linear problem is analyzed in a previous paper, see [5].

In the paper [12], it is presented a fourth-order model with applications to the study of the position of a loaded elastic beam, problem where the boundary conditions are crucial to specify the type of support imposed at the endpoints of the beam. Nonlocal and multipoint boundary conditions are also important in this problem, since they allow to control different magnitudes (position, angle, etc.) of the beam at different points. For the study of some nonlinear boundary conditions of interest in the representation of feedback controls, see [12], which also addresses to other related references.

In [16], the authors consider differential equations of arbitrary order with any number of nonlocal boundary conditions, showing some particular cases such as the equation of Euler–Bernoulli for the elastic curve of a deflected beam (length 1), represented by the fourth-order equation $u^{(4)}(t) = g(t)f(t, u(t))$, $t \in (0, 1)$, where the right-hand side represents the load along the beam, adding the nonlocal boundary conditions $u(0) = \beta_1[u],\ u'(0) = \beta_2[u],\ u(1) = \beta_3[u],\ u''(1) + \beta_4[u] = 0$, involving linear functionals β_j. These conditions are explained by the existence of a controller of the position at $t = 0$ whose effect depends on the measurements of displacement along the beam, also a controller of angular attitude and, at the right endpoint, the position and the bending moment are regulated. For other boundary value problems for an elastic beam equation, see [21] and also [17], where a boundary value problem is studied for an elastic beam equation with a corner which describes the deformation of the beam under a certain force. Ref. [6] presents different nonlocal boundary conditions which include a regulator of the displacement or the angular attitude, or even a control on the bending moment at an endpoint of the beam.

In [1], three-point boundary value problems are considered for a nonlinear Langevin equation involving two fractional orders, equation whose different variants are useful to explain the behavior of phenomena in environments subject to fluctuation, motion of particles, or dynamical processes, for instance. Ref. [4] is focused on the nonlocal problem appearing in the study of the magnetic confinement of a plasma in a current-carrying Stellarator device.

Concerning nonlocal initial problems for parabolic equations, [2] analyzes coupled reaction–diffusion systems, presenting some applications to epidemiology and ecology. Besides, the Refs. [10,14] consider a reaction–diffusion model of plankton allelopathy on PDEs with nonlocal delays under initial conditions and homogeneous Neumann boundary conditions, and [13] provides some existence results for nonlocal boundary value problems for third order nonlinear differential equations, with applications to nano boundary layer fluid flows over stretching surfaces. In [22], the application of fixed point results allows to prove the existence of multiple non-negative solutions to nonlocal boundary value problems for third-order advanced differential equations. On the other hand, [23] is devoted to the study of the existence of solution to general nonlocal boundary value problems for nonlinear differential equations by establishing connections to linear Sturm–Liouville problems, and [24] considers nonlocal boundary value problems for second-order differential equations with a piecewise constant functional dependence. The review paper [25] compiles some results on the existence of solutions for nonlinear boundary value problems for second-order differential equations based on the method of the upper and lower solutions and the work [26] analyzes the continuity and differentiability properties of the solutions to impulsive differential equations with respect to factors such as initial conditions and impulsive perturbations, showing some applications.

In the literature, several references are devoted to the study of the existence of solution for differential equations with multipoint boundary value conditions and also to the analysis of their spectral properties. For instance, in [27], the existence and uniqueness of solutions for problem

$$x'(t) = f(t, x(t)) + e(t), \quad t \in (0, 1), \qquad \sum_{j=1}^{m} A_j x(\eta_j) = 0,$$

is proved, where f is a Carathéodory function in $\mathbb{R}^n$, A_j are constant $n \times n$ matrices and η_j is a finite increasing sequence in $(0, 1)$. In [28], the authors consider multipoint boundary value problems for p-Laplacian equations and study the existence of multiple positive solutions by applying fixed point results for operators in a cone. The paper [29] also provides results on multipoint boundary value problems for differential equations of higher order. Moreover, some results on linear differential equations with singularities are included in [30]. Ref. [31] shows the applicability of fixed point theory to provide sufficient conditions for the existence of countably many positive solutions for singular multipoint boundary value problems.

The work [32] proves some existence and uniqueness results for multipoint boundary value problems for nonlinear ordinary differential equations. For spectral properties related to multipoint boundary value problems for p-Laplacian equations, we refer to [33,34]. See also [35,36] for the case of ordinary differential equations. On the other hand, the study of the existence of solutions for multipoint boundary conditions for second-order differential equations can be found, for example, in [37–43].

With respect to the solvability of multipoint boundary value problems for ordinary differential equations, we also mention [44], where the explicit solution is obtained for the case of first-order linear differential equations by calculating Green's function.

The aim of this work is to study the existence of solution for a class of multipoint boundary value problems of Neumann type for linear functional differential equations. First, we solve a similar problem for ordinary differential equations and, hence, we apply the results obtained and fixed point theory to find sufficient conditions for the existence of solution to