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# Existence and structure of solutions for a class of optimal control systems with discounting arising in economic dynamics

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ABSTRACT

We study the existence and a turnpike property of solutions of a discrete-time control system with discounting and with a compact metric space of states. In our recent work for this discrete-time optimal control system without discounting we establish the turnpike property and show that it is stable under perturbations of an objective function. In the present paper we show that this turnpike property together with its stability also hold for the system with discounting.



## 1. Introduction

Let $(X, \rho)$ be a compact metric space and $\Omega$ be a nonempty closed subset of $X \times X$.

A sequence $\{x_t\}_{t=0}^{\infty} \subset X$ is called a program if $(x_t, x_{t+1}) \in \Omega$ for all integers $t \geq 0$. A sequence $\{x_t\}_{t=T_1}^{T_2} \subset X$ where integers $T_1, T_2$ satisfy $0 \leq T_1 < T_2$ is called a program if $(x_t, x_{t+1}) \in \Omega$ for all integers $t \in [T_1, T_2 - 1]$.

In this paper we consider the problem

$$\sum_{i=0}^{T-1} v_i(x_i, x_{i+1}) \to \max \tag{P}$$

$$\text{s.t. } \{(x_i, x_{i+1})\}_{i=0}^{T-1} \subset \Omega, \quad x_0 = z,$$

where $T$ is a natural number, $z \in X$ and $v_t : \Omega \to R^1$, $t = 0, \ldots, T - 1$ are bounded functions. This discrete-time optimal control system describes a general model of economic dynamics [1–9], where the set $X$ is the space of states, $v_t$ is a utility function and $v_t(x_t, x_{t+1})$ evaluates consumption at moment $t$. The interest in discrete-time optimal problems of type (P) also stems from the study of various optimization problems which can be reduced to it, e.g., tracking problems in engineering [10], the study of Frenkel–Kontorova model related to dislocations in one-dimensional crystals [11,12] and the analysis of a long slender bar of a polymeric material under tension in [13]. Optimization problems of the type (P) with $\Omega = X \times X$ were considered in [14,15,6].

We are interested in a turnpike property of the approximate solutions of (P) which is independent of the length of the interval $T$, for all sufficiently large intervals. To have this property means, roughly speaking, that the solutions of the optimal control problems are determined mainly by the cost functions $v_t$, and are essentially independent of $T$ and $z$. Turnpike properties are well known in mathematical economics. The term was first coined by Samuelson in 1948 (see [5]) where he showed that an efficient expanding economy would spend most of the time in the vicinity of a balanced equilibrium path (also called a von Neumann path).

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It should be mentioned that the study of the existence and the structure of solutions of optimal control problems defined on infinite intervals and on sufficiently large intervals has recently been a rapidly growing area of research. See, for example, [16–27] and the references mentioned therein.

In the classical turnpike theory [1–4] the space $X$ is a compact convex subset of a finite-dimensional Euclidean space, the set $\Omega$ is convex and $v_t = v$ for all integers $t \geq 0$, where the function $v$ is strictly concave. Under these assumptions the turnpike property can be established and the turnpike $\bar{x}$ is a unique solution of the maximization problem $v(x, x) \to \max$, $(x, x) \in \Omega$. In this situation it is shown that for each program $\{x_t\}_{t=0}^{\infty}$ either the sequence $\{\sum_{t=0}^{T-1} v(x_t, x_{t+1}) - Tv(\bar{x}, \bar{x})\}_{T=1}^{\infty}$ is bounded (in this case the program $\{x_t\}_{t=0}^{\infty}$ is called $(v)$-good) or it diverges to $-\infty$. Moreover, it is also established that any $(v)$-good program converges to the turnpike $\bar{x}$. In the sequel this property is called as the asymptotic turnpike property.

In [7,8] we studied problem (P) with $v_t = v$ for all integers $t \geq 0$, where $v : \Omega \to R^1$ is a bounded function, and showed that the turnpike property follows from the asymptotic turnpike property. More precisely, we assumed that any $(v)$-good program converges to a unique solution $\bar{x}$ of the problem $v(x, x) \to \max$, $(x, x) \in \Omega$ and showed that the turnpike property holds and $\bar{x}$ is the turnpike. Note that we do not use convexity (concavity) assumptions. In [9] we showed that the turnpike property established in [7,8] is stable under perturbations of the objective function $v$.

In the present paper we show that the turnpike property together with its stability established in [7–9] also hold for the model with discounting.

In particular, the turnpike property holds for solutions of problem (P) with $v_t = \beta^t u$, $t = 0, 1, \ldots$, where $\beta \in (0, 1]$ and $1 - \beta$ is small enough and $u$ is close enough to $v$ in the topology of the uniform convergence on $\Omega$, where $v$ is a bounded function on $\Omega$ which possesses the asymptotic turnpike property.

Let $(X, \rho)$ be a compact metric space and $\Omega$ be a nonempty closed subset of $X \times X$. Denote by $\mathcal{M}$ the set of all bounded functions $u : \Omega \to R^1$. For each $w \in \mathcal{M}$ set

$$\|w\| = \sup\{|w(x, y)| : (x, y) \in \Omega\}. \tag{1.1}$$

For each $x, y \in X$, each integer $T \geq 1$ and each $u \in \mathcal{M}$ set

$$\sigma(u, T, x) = \sup\left\{\sum_{i=0}^{T-1} u(x_i, x_{i+1}) : \{x_i\}_{i=0}^{T} \text{ is a program and } x_0 = x\right\}, \tag{1.2}$$

$$\sigma(u, T, x, y) = \sup\left\{\sum_{i=0}^{T-1} u(x_i, x_{i+1}) : \{x_i\}_{i=0}^{T} \text{ is a program and } x_0 = x,\ x_T = y\right\}, \tag{1.3}$$

$$\sigma(u, T) = \sup\left\{\sum_{i=0}^{T-1} u(x_i, x_{i+1}) : \{x_i\}_{i=0}^{T} \text{ is a program}\right\}. \tag{1.4}$$

(Here we use the convention that the supremum of an empty set is $-\infty$ and the sum over empty set is zero.)

For each $x, y \in X$, each pair of integers $T_1, T_2$ satisfying $0 \leq T_1 < T_2$ and each sequence $\{u_t\}_{t=T_1}^{T_2-1} \subset \mathcal{M}$ set

$$\sigma(\{u_t\}_{t=T_1}^{T_2-1}, T_1, T_2, x) = \sup\left\{\sum_{t=T_1}^{T_2-1} u_t(x_t, x_{t+1}) : \{x_t\}_{t=T_1}^{T_2} \text{ is a program and } x_{T_1} = x\right\}, \tag{1.5}$$

$$\sigma(\{u_t\}_{t=T_1}^{T_2-1}, T_1, T_2, x, y) = \sup\left\{\sum_{i=T_1}^{T_2-1} u_t(x_t, x_{t+1}) : \{x_t\}_{t=T_1}^{T_2} \text{ is a program and } x_{T_1} = x,\ x_{T_2} = y\right\}, \tag{1.6}$$

$$\sigma(\{u_t\}_{t=T_1}^{T_2-1}, T_1, T_2) = \sup\left\{\sum_{t=T_1}^{T_2-1} u_t(x_t, x_{t+1}) : \{x_t\}_{t=T_1}^{T_2} \text{ is a program}\right\}. \tag{1.7}$$

Assume that $v \in \mathcal{M}$ is an upper semicontinuous function. Since in [7,8] we assume that objective functions are defined on the set $X \times X$ in order to apply their results we set $v(x, y) = -\|v\| - 1$ for all $(x, y) \in (X \times X) \setminus \Omega$.

We suppose that there exist $\bar{x} \in X$ and a constant $\bar{c} > 0$ such that the following assumptions hold.

(A1) $(\bar{x}, \bar{x})$ is an interior point of $\Omega$ (there is $\epsilon > 0$ such that $\{(x, y) \in X \times X : \rho(x, \bar{x}),\ \rho(y, \bar{x}) \leq \epsilon\} \subset \Omega$) and $v$ is continuous at $(\bar{x}, \bar{x})$.

(A2) $\sigma(v, T) \leq Tv(\bar{x}, \bar{x}) + \bar{c}$ for all integers $T \geq 1$.

It is easy to see that for each natural number $T$ and each program $\{x_t\}_{t=0}^{T}$

$$\sum_{t=0}^{T-1} v(x_t, x_{t+1}) \leq \sigma(v, T) \leq Tv(\bar{x}, \bar{x}) + \bar{c}. \tag{1.8}$$

Inequality (1.8) implies the following result.