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Deformation of surfaces in three-dimensional space induced by means of integrable systems

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A B S T R A C T

The correspondence between different versions of the Gauss–Weingarten equation is investigated. The compatibility condition for one version of the Gauss–Weingarten equation gives the Gauss–Mainardi–Codazzi system. A deformation of the surface is postulated which takes the same form as the original system but contains an evolution parameter. The compatibility condition of this new augmented system gives the deformed Gauss–Mainardi–Codazzi system. A Lax representation in terms of a spectral parameter associated with the deformed system is established. Several important examples of integrable equations based on the deformed system are then obtained. It is shown that the Gauss–Mainardi–Codazzi system can be obtained as a type of reduction of the self-dual Yang–Mills equations.

1. Introduction

There are a great many phenomena in nature which make use of the concept of a surface in formulating a realistic and useful model which accounts for the observations and properties which are of interest to investigate [1]. Quite frequently, it is required by the nature of the circumstances that these surfaces evolve over the course of time, as opposed to remaining completely static. An example of the former would be a propagating shock front, and of the latter, a surface formed by the surface tension at the interface between two different liquids. In the course of the development of this kind of theory, there occurs as a consequence of the method or process used the appearance of various kinds of nonlinear partial differential equations in a natural way. As a result, the study of all aspects of these equations becomes an important topic in itself, and becomes linked to the study of the evolution problem. In the larger picture, there results an interaction between the areas which concern the differential geometry of surfaces [2] and nonlinear partial differential equations which arise in the course of this work [3–5]. This kind of interaction has been of mutual benefit to the development of these subjects.

Consequently, many nonlinear phenomena in physics, which can be described by various kinds of partial differential equation, are also closely related to the evolution of surfaces with respect to an evolution parameter such as time [6]. Moreover, it has been found that these types of nonlinear equations possess solitary wave solutions [7,8]. Thus, there are a great number of links between many diverse areas and this is largely based on the fact that a great many of the local properties of surfaces can be expressed in the form of nonlinear partial differential equations. For example, two equations which have played a particularly important role in the development of the subject are the sine-Gordon and Liouville equations, respectively. These equations, especially the former, have also played a prominent role in the development of Bäcklund transformations. In fact, a generic method for the description of soliton interaction has its roots in a type of transformation originally introduced by Bäcklund to generate pseudospherical surfaces [9,10]. The sine-Gordon equation was generated in the nineteenth century from the Gauss–Mainardi–Codazzi system for pseudo-spherical surfaces. This equation was subsequently rederived independently by both Enneper and Bonnet in a similar way. A purely geometric construction for pseudospheri-

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cal surfaces was reformulated later as a transformation by Bianchi. The interrelationship between deformations of surfaces and integrable systems in $2+1$ dimensions has been discussed by many researchers [11,12].

The objective here is to investigate the deformation of surfaces and the relationship of this topic to the study of various aspects of integrable systems in various dimensions. It will be seen that many integrable $(2+1)$-dimensional nonlinear partial differential equations can be obtained from the $(2+1)$-dimensional Gauss–Mainardi–Codazzi equation, which can be interpreted as describing the deformation, or motion, of a surface. It is hoped that the discussion will benefit from the new accompanying proofs.

A particularly remarkable example which is to be studied is that of the reduction of the self-dual Yang–Mills equation to Gauss–Mainardi–Codazzi form. Yang–Mills systems have numerous applications in particle physics. It may be said that the self-dual Yang–Mills system appears to be a universal integrable system from which many other integrable equations can be obtained by symmetry reductions and specification of the Lie algebra. In fact, it has been conjectured by Ward [13] that all integrable $(1+1)$-dimensional nonlinear differential equations may be obtained by reduction directly from the self-dual Yang–Mills equations. In fact, many soliton equations in $(2+1)$ dimensions have been found as reductions of the same self-dual system, and some new ones appear here. Finally, it will be shown that the linear systems introduced here give rise to a number of integrable equations which are well known and of interest. These equations are developed as a result of specific reductions of the Gauss–Mainardi–Codazzi system.

2. Surface theory and the Gauss–Weingarten equation

Let M be a smooth manifold or surface in $\mathbb{R}^3$ with a local coordinate system specified by (x, y). Let $\mathbf{r} = \mathbf{r}(x, y)$ denote the position vector of a generic point P on M in $\mathbb{R}^3$. The vectors $\mathbf{r}_x$ and $\mathbf{r}_y$ are tangential to M at P, and at points where they are linearly independent

$$\mathbf{n} = \frac{\mathbf{r}_x \times \mathbf{r}_y}{|\mathbf{r}_x \times \mathbf{r}_y|}, \tag{2.1}$$

determine a unit normal to M. The first and second fundamental forms of this surface are given by

$$\mathrm{I} = d\mathbf{r}^2 = E\,dx^2 + 2F\,dxdy + G\,dy^2, \tag{2.2}$$

$$\mathrm{II} = d\mathbf{r} \cdot \mathbf{n} = L\,dx^2 + 2M\,dxdy + N\,dy^2, \tag{2.3}$$

where the coefficient terms are defined to be,

$$E = \mathbf{r}_x^2, \qquad F = \mathbf{r}_x \cdot \mathbf{r}_y, \qquad G = \mathbf{r}_y^2, \tag{2.4}$$

$$L = \mathbf{r}_{xx} \cdot \mathbf{n}, \qquad M = \mathbf{r}_{yx} \cdot \mathbf{n}, \qquad N = \mathbf{r}_{yy} \cdot \mathbf{n}. \tag{2.5}$$

An important classical result due to Bonnet states that the sextuplet $\{E, F, G, L, M, N\}$ determines M up to its position in space. There is a third fundamental form which does not depend on the choice of $\mathbf{n}$ and does not contain much beyond what is prescribed by (2.2)–(2.3) since it is expressible in terms of I and II as

$$\mathrm{III} = d\mathbf{n} \cdot d\mathbf{n} = 2H \cdot \mathrm{II} - K \cdot \mathrm{I}. \tag{2.6}$$

In (2.6), K, H are the Gaussian and mean curvatures of M, respectively.

The Gauss equations associated with M are

$$\mathbf{r}_{xx} = \Gamma_{11}^1 \mathbf{r}_x + \Gamma_{11}^2 \mathbf{r}_y + L\mathbf{n}, \qquad \mathbf{r}_{xy} = \Gamma_{12}^1 \mathbf{r}_x + \Gamma_{12}^2 \mathbf{r}_y + M\mathbf{n}, \qquad \mathbf{r}_{yy} = \Gamma_{22}^1 \mathbf{r}_x + \Gamma_{22}^2 \mathbf{r}_y + N\mathbf{n}, \tag{2.7}$$

while the Weingarten equations are given as

$$\mathbf{n}_x = P_1^1\,\mathbf{r}_x + P_1^2 \mathbf{r}_y, \qquad \mathbf{n}_y = P_2^1 \mathbf{r}_x + P_2^2 \mathbf{r}_y. \tag{2.8}$$

The ten coefficient functions in systems (2.7)–(2.8) are given in terms of the sextuplet $\{E, F, G, L, M, N\}$ as follows,

$$\begin{gathered}\Gamma_{11}^1 = \frac{1}{2g}(GE_x - 2FF_x + FE_y), \qquad \Gamma_{11}^2 = \frac{1}{2g}(2EF_x - EE_y - FE_x),\\ \Gamma_{12}^1 = \frac{1}{2g}(GE_y - FG_x), \qquad \Gamma_{12}^2 = \frac{1}{2g}(EG_x - FE_y),\\ \Gamma_{22}^1 = \frac{1}{2g}(2GF_y - GG_x - FG_y), \qquad \Gamma_{22}^2 = \frac{1}{2g}(EG_y - 2FF_y + FG_x),\end{gathered} \tag{2.9}$$

$$P_1^1 = \frac{MF - LG}{g}, \qquad P_1^2 = \frac{LF - ME}{g}, \qquad P_2^1 = \frac{NF - MG}{g}, \qquad P_2^2 = \frac{MF - NE}{g}, \tag{2.10}$$

where

$$g = |\mathbf{r}_x \times \mathbf{r}_y|^2 = EG - F^2. \tag{2.11}$$