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Asymptotically linear Hamiltonian system

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ABSTRACT

We investigate the multiplicity of solutions for the Hamiltonian system with some asymptotically linear conditions. We get a theorem which shows the existence of at least three 2π-periodic solutions for the asymptotically linear Hamiltonian system. We obtain this result by the variational reduction method which reduces the infinite dimensional problem to the finite dimensional one. We also use the critical point theory and the variational method.

1. Introduction and statement of main result

Let $G(t, z(t))$ be a C^2 function defined on $R^1 \times R^{2n}$ which is 2π-periodic with respect to the first variable t. In this paper we investigate the number of 2π-periodic solutions of the following Hamiltonian system

$$\dot{p(t)} = -G_q(t, p(t), q(t)), \quad (1.1)$$
$$\dot{q(t)} = G_p(t, p(t), q(t)),$$

where $p, q \in R^n$, $z = (p, q)$. Let J be the standard symplectic structure on R^{2n}, i.e.,

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix},$$

where I_n is the $n \times n$ identity matrix. Then (1.1) can be rewritten as

$$-J\dot{z} = G_z(t, z(t)) \quad (1.2)$$

where $\dot{z} = \frac{dz}{dt}$ and G_z is the gradient of G. We assume that $G \in C^2(R^1 \times R^{2n}, R^1)$ satisfies the following asymptotically linear conditions:

(G1) $G(t, z(t)) = 0(|z|^2)$ as $|z| \to 0$, $G(t, \theta) = 0$, $G_z(t, \theta) = \theta$, where $\theta = (0, \ldots, 0)$,

(G2) There exist constants α, β (without loss of generality, we may assume $\alpha, \beta \notin Z$) such that

$$\alpha I \le d_z^2 G(t, z) \le \beta I, \quad \forall (t, z) \in R^1 \times R^{2n}.$$

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(G3) Let j_1 be a integer within $[\alpha, \beta]$ such that

$$j_1 - 1 < \alpha < d_z^2 G(t, 0) = \lim_{|z| \to 0} \frac{G_z(t, z) \cdot z}{|z|^2} < j_1,$$

(G4) $\lim_{|z| \to \infty} \frac{G_z(t,z) \cdot z}{|z|^2}$ exists and there exists $j_2 = j_1 + 1$ which satisfies

$$j_1 < d_z^2 G(t, \infty) = \lim_{|z| \to \infty} \frac{G_z(t, z) \cdot z}{|z|^2} < \beta < j_2,$$

(G5) G is 2π-periodic with respect to t.

We are looking for the weak solutions of (1.1) with the conditions (G1)–(G5). The 2π-periodic weak solution $z = (p, q) \in E$ of (1.1) satisfies

$$\int_0^{2\pi} \dot{z} \cdot w - J(G_z(t, z(t))) \cdot Jw dt = 0 \quad \text{for all } w \in E,$$

i.e.,

$$\int_0^{2\pi} [(\dot{p} + G_q(t, z(t))) \cdot \psi - (\dot{q} - G_p(t, z(t))) \cdot \phi] dt = 0 \quad \text{for all } \zeta = (\phi, \psi) \in E,$$

where E is introduced in Section 2. By Lemma 2.1 in Section 2, the weak solutions of (1.1) coincide with the critical points of the functional

$$\begin{aligned} f(z) &= \frac{1}{2} \int_0^{2\pi} (-J\dot{z}) \cdot z dt - \int_0^{2\pi} G(t, z(t)) dt \\ &= \int_0^{2\pi} p\dot{q} dt - \int_0^{2\pi} G(t, z(t)) dt. \end{aligned} \tag{1.3}$$

Chang proved in [1] that if $G \in C^2(R^1 \times R^{2n}, R^1)$ satisfies the conditions (G2), (G5) and the following additional conditions:

(G3)′ Let $j_0, j_0 + 1, \ldots,$ and j_1 be all integers within $[\alpha, \beta]$ (without loss of generality, we may assume $\alpha, \beta \notin Z$) such that $j_0 - 1 < \alpha < j_0 < j_1 < \beta < j_1 + 1 = j_2$. Suppose that there exist $\gamma > 0$ and $\tau > 0$ such that $j_1 < \gamma < \beta$ and

$$G(t, z) \geq \frac{1}{2} \gamma \|z\|_{L^2}^2 - \tau, \quad \forall (t, z) \in R^1 \times R^{2n}.$$

(G4)′ $G_z(t, \theta) = \theta$ and $j \in [j_0, j_1) \cap Z$ such that

$$jI < d_z^2 G(t, \theta) < (j + 1)I, \quad \forall t \in R^1,$$

then (1.1) has at least two nontrivial 2π-periodic weak solutions. Jung and Choi proved in [2] that if G satisfies the following conditions:

(G1)″ $G : R^{2n} \to R$ is C^1 with $G(\theta) = 0$.

(G2)″ There exists $h \in N$ such that

$$h < \liminf_{|z| \to \infty} \frac{G'(z) \cdot z}{|z|^2} < h + 1.$$

(G3)″ There exists $m \in N$ such that

$$h + 2m < \liminf_{|z| \to 0} \frac{G'(z) \cdot z}{|z|^2} < h + 2m + 1,$$

or

$$h - 2m - 1 < \limsup_{|z| \to 0} \frac{G'(z) \cdot z}{|z|^2} < h - 2m.$$

(G4)″ There exists an integer Γ such that $\Gamma \leq \frac{G'(z) \cdot z}{|z|^2} \leq \Gamma + 1$,

then (1.1) has at least m weak solutions, which are geometrically distinct and nonconstant.

Our main result is the following:

Theorem 1.1. *Assume that G satisfies the conditions* (G1)–(G5). *Then system* (1.1) *has at least three 2π-periodic solutions.*