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Multiple solutions for the nonhomogeneous Kirchhoff equation on $\mathbf{R}^N$ ☆

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ABSTRACT

In this paper we study the following nonhomogeneous Kirchhoff equation

$$-\left(a+b\int_{\mathbf{R}^N}|\nabla u|^2dx\right)\Delta u+V(x)u=f(x,u)+h(x),\quad \text{in } \mathbf{R}^N,$$

where f satisfies the Ambrosetti–Rabinowitz type condition. Under appropriate assumptions on V, f and h, the existence of multiple solutions is proved by using the Ekeland's variational principle and the Mountain Pass Theorem in critical point theory.

1. Introduction and main results

This paper was motivated by some works that had appeared in recent years concerning the following Kirchhoff-type problem

$$-\left(a+b\int_{\Omega}|\nabla u|^2dx\right)\Delta u=g(x,u),\quad \text{in } \Omega, \tag{1}$$

where $\Omega\subset\mathbf{R}^N$ is a smooth domain, $a>0$, $b\geq 0$ and u satisfies some initial or boundary condition.

The problem (1) is related to the stationary analogue of the Kirchhoff equation

$$u_{tt}-\left(a+b\int_{\Omega}|\nabla_x u|^2dx\right)\Delta_x u=g(x,u) \tag{2}$$

which was proposed by Kirchhoff in 1883 (see [1]) as a generalization of the well-known d'Alembert's wave equation

$$\rho\frac{\partial^2 u}{\partial t^2}-\left(\frac{P_0}{h}+\frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2dx\right)\frac{\partial^2 u}{\partial x^2}=g(x,u)$$

for free vibrations of elastic strings. Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations. Here, L is the length of the string, h is the area of the cross section, E is the Young modulus of the material, ρ is the mass density and P_0 is the initial tension.

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In [2], it was pointed out that the problem (2) models several physical systems, where u describes a process which depends on the average of itself. Nonlocal effect also finds its applications in biological systems. A parabolic version of Eq. (1) can, in theory, be used to describe the growth and movement of a particular species. The movement, modeled by the integral term, is assumed to be dependent on the energy of the entire system with u being its population density. Alternatively, the movement of a particular species may be subject to the total population density within the domain (for instance, the spreading of bacteria) which gives rise to equations of the type $u_t - a(\int_\Omega u dx)\Delta u = h$. Some early classical studies of Kirchhoff's equation were those of Bernstein [3] and Pohožaev [4]. However, Eq. (2) received great attention only after that Lions [5] proposed an abstract framework for the problem. Some interesting results for problem (2) can be found in [6–8] and the references therein.

Some interesting studies by variational methods can be found in [2,9–22] for Kirchhoff-type problem (1), they consider in a bounded domain of $\Omega \subset \mathbf{R}^N$. Very recently, some authors had studied the Kirchhoff equation on the whole space $\mathbf{R}^N$. Jin and Wu [23] obtained the existence of infinitely many radial solutions for problem (1) in $\mathbf{R}^N$ using the Fountain Theorem. In [24], Wu gets four new existence results for nontrivial solutions and a sequence of high energy solutions for problem (1) in $\mathbf{R}^N$ which was obtained by using the Symmetric Mountain Pass Theorem. In [25], Azzollini, d'Avenia and Pomponio get a multiplicity result concerning the critical points of a class of functionals involving local and nonlocal nonlinearities, then they apply their result to the nonlinear elliptic Kirchhoff equation (1) in $\mathbf{R}^N$ assuming that the local nonlinearity satisfies the general hypotheses introduced by Berestycki and Lions [26]. He and Zou [27] study the existence, multiplicity and concentration behavior of positive solutions for the nonlinear Kirchhoff type problem. They relate the number of solutions with the topology of the set. Recently, Nie and Wu [28] have studied a Schrödinger–Kirchhoff-type equation with radial potential, and multiplicity of nontrivial solutions were obtained by the Mountain Pass Theorem and the symmetric Mountain Pass Theorem. In [29], Alves and Figueiredo study a periodic Kirchhoff equation in $\mathbf{R}^N$, they get the nontrivial solution when the nonlinearity is in subcritical case and critical case. Liu and He [30] get multiplicity of high energy solutions for superlinear Kirchhoff equations in $\mathbf{R}^3$.

In the same spirit of [25,23,24,27–30], we study a nonhomogeneous Kirchhoff equation on the whole space $\mathbf{R}^N$, namely we consider the problem

$$-\left(a + b\int_{\mathbf{R}^N} |\nabla u|^2 dx\right)\Delta u + V(x)u = f(x,u) + h(x), \quad \text{in } \mathbf{R}^N. \tag{3}$$

In the sense, the problem turns out to be a generalization of the well know nonhomogeneous Schrödinger equation:

$$-\Delta u + V(x)u = f(x,u) + h(x), \quad \text{in } \mathbf{R}^N.$$

In this paper, we are interested in looking for multiple solutions of the problem (3). Unlike [23,28,29], we consider a Bartsch-type potential. To this end, we make the following assumptions.

(v1) $V \in C(\mathbf{R}^N, \mathbf{R})$ satisfies $\inf_{x\in\mathbf{R}^N} V(x) \geq a_1 > 0$, where $a_1 > 0$ is a constant. Moreover, for every $M > 0$, meas $\left(\{x \in \mathbf{R}^N : V(x) \leq M\}\right) < \infty$, where meas denote the Lebesgue measure in $\mathbf{R}^N$.

(f1) $f \in C(\mathbf{R}^N \times \mathbf{R}, \mathbf{R})$ and, for some $2 < p < 2^* = 2N/(N-2)$, $a_2 > 0$,

$$|f(x,z)| \leq a_2(1 + |z|^{p-1}),$$

for a.e. $x \in \mathbf{R}^N$ and all $z \in \mathbf{R}$.

(f2) There exists $\mu > 4$ such that

$$\mu F(x,z) := \mu \int_0^z f(x,y)dy \leq z f(x,z),$$

for every $x \in \mathbf{R}^N$ and all $z \in \mathbf{R}$.

(f3) $f(x,z)/z \to 0$, as $z \to 0$, uniformly for $x \in \mathbf{R}^N$.

(f4)

$$\inf_{x\in\mathbf{R}^N, |z|=1} F(x,z) > 0.$$

Before stating our main results, we give several notations. Define the function space

$$H^1(\mathbf{R}^N) := \left\{u \in L^2(\mathbf{R}^N) : |\nabla u| \in L^2(\mathbf{R}^N)\right\}$$

with the usual norm

$$\|u\|_{H^1} := \left(\int_{\mathbf{R}^N} \left(|\nabla u|^2 + u^2\right) dx\right)^{\frac{1}{2}}.$$

Let

$$E := \left\{u \in H^1(\mathbf{R}^N) : \int_{\mathbf{R}^N} \left(|\nabla u|^2 + V(x)u^2\right) dx < \infty\right\}.$$