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Quasi self-adjointness of a class of third order nonlinear dispersive equations

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ABSTRACT

In this paper we find the conditions of *self-adjointness* and *quasi self-adjointness* for a class of third order PDEs. As an application, in some cases, we get conservation laws.

1. Introduction

We consider the following class of nonlinear prototype equations:

$$u_t + f_1(u, u_x) + u_{xx}f_2(u, u_x) + (u_{xx}f_3(u, u_x))_x = 0. \tag{1}$$

Aside from the KdV type equations and modified KdV equations (see e.g. [1–4]), some known equations of the Mathematical Physics fall in this class, as Harry–Dym [5] and modified Harry–Dym equations [6], the Chou–Qu–Huber equation [7,8], and $K(m, n)$ equations [9] with their specializations and generalizations [10–14]. Moreover the following new family of nonlinear dispersive equations falls in this class:

$$u_t + (u^n)_x + \frac{1}{b}(u^a(u^b)_{xx})_x = 0, \tag{2}$$

with $n > \max(1, a - b)$ and $b > 0$. This class is denoted in the literature as $C_1(n, a + b)$ and has been recently introduced by Rosenau [15]. Eq. (2) was extended in [16] to the following more general equation of the form

$$u^{m-1}u_t + \alpha(u^n)_x + \beta(u^a(u^b)_{xx})_x = 0, \tag{3}$$

where m, n, a, α and β are constants. This equation also can be reduced to the equation of the class (1).

Recently, following a procedure introduced in [17–20], several papers have been devoted to search for self-adjoint equations. This procedure, in fact, can be applied not only to classes of single differential equations [21–24], of any order but to the systems [25,26] where the number of equations is equal to the number of dependent variables.

In this paper after recalling some concepts introduced in [17–20] we consider the equations of the type (1) and look for a *quasi self-adjoint classification* with respect to the functions f_1, f_2 and f_3.

Noether's theorem, as known, shows a connection between symmetries of differential equations and conservation laws, provided that these equations possess a variational formulation. However, this restriction, reduces the possibilities of Noether's theorem application as the Lagrangians exist only in special cases.

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This kind of problems has been overcame in the recent papers [17,19] where starting from the new concept of formal Lagrangian has been proved a theorem that establishes a connection between their symmetries and conservation laws even if Noether's theorem is not applicable. Here taking these results into account, we show how to obtain conservation laws for equations belonging to the class (1).

The outline of the paper is as follows. In Section 2 after some preliminaries we write the adjoint equations to (1). In Section 3 we get the conditions of *self-adjointness* and *quasi self-adjointness*; moreover after having solved them we characterize subclasses of *self-adjoint* equations and of *quasi self-adjoint* equations belonging to (1). In Section 4, by using the theorem proved in [19], conservation laws are found for some quasi self-adjoint equations of the class (1). Conclusions are given in Section 5.

2. Adjoint equations

According to [17] (see also [19]) we introduce the following function

$$\mathcal{L} = v[u_t + f_1(u, u_x) + u_{xx}f_2(u, u_x) + (u_{xx}f_3(u, u_x))_x], \tag{4}$$

which will be called *formal Lagrangian* [17]. Following [17] the adjoint equation to (1) has the form

$$\frac{\delta \mathcal{L}}{\delta u} = 0, \tag{5}$$

where

$$\frac{\delta \mathcal{L}}{\delta u} = \frac{\partial \mathcal{L}}{\partial u} - D_t\left(\frac{\partial \mathcal{L}}{\partial u_t}\right) - D_x\left(\frac{\partial \mathcal{L}}{\partial u_x}\right) + D_x^2\left(\frac{\partial \mathcal{L}}{\partial u_{xx}}\right) - D_x^3\left(\frac{\partial \mathcal{L}}{\partial u_{xxx}}\right). \tag{6}$$

After observing that

$$v\,(u_{xx}f_3(u, u_x))_x = (v\,u_{xx}f_3(u, u_x))_x - v_x\,u_{xx}f_3(u, u_x),$$

we can write Eq. (4) in the following equivalent second-order form

$$\mathcal{L} = v\,u_t + v\,f_1(u, u_x) + v\,u_{xx}f_2(u, u_x) - v_x\,u_{xx}f_3(u, u_x). \tag{7}$$

Applying Eqs. (5), (7), (6) the adjoint equation to (1) reads

$$F^* \equiv -v_t - f_3 v_{xxx} + \left(-2\,u_x f_{3u} - u_{xx}f_{3u_x} + f_2\right) v_{xx} + (-f_{1u_x} + u_{xx}f_{2u_x} - u_x u_{xx} f_{3uu_x} + 2\,u_x f_{2u}$$
$$- 2\,u_{xx}f_{3u} - u_x^2 f_{3uu})v_x + (-u_{xx}\,f_{1u_xu_x} - u_x\,f_{1uu_x} + 2\,u_{xx}f_{2u} + u_x\,u_{xx}f_{2uu_x} + u_x^2 f_{2uu} + f_{1u})v = 0.$$

For further applications, we remark the following statement, proved in [18,19], that provides a link between the symmetry generators of an equation and its adjoint equation.

Theorem 1. *Any symmetry (Lie point, Lie–Bäcklund, nonlocal symmetry , . . .)*

$$X = \xi^i(x, u, u_{(1)}, \ldots)\frac{\partial}{\partial x_i} + \eta(x, u, u_{(1)}, \ldots)\frac{\partial}{\partial u}$$

of an equation

$$F(x, u, u_{(1)}, \ldots, u_{(s)}) = 0, \tag{8}$$

with n independent variables $x = (x_1, \ldots, x_n)$ *and the dependent variable u is inherited by the adjoint equation. Specifically the operator*

$$Y = \xi^i\frac{\partial}{\partial x_i} + \eta\frac{\partial}{\partial u} + \eta_*\frac{\partial}{\partial v},$$

with an appropriately chosen coefficient η_* *is admitted by the system consisting of Eq. (8) and its adjoint equation*

$$F^*(x, u, v, \ldots, u_{(s)}, v_{(s)}) \equiv \frac{\delta(vF)}{\delta u} = 0.$$

3. Conditions for quasi self-adjointness and self-adjointness

We recall the following definitions [17,19,20,25].

Definition 1. A nonlinear differential equation is said to be self-adjoint if its adjoint equation becomes equivalent to the original equation after the substitution

$$v = u.$$