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On the existence of an exponential attractor for a planar shear flow with the Tresca friction condition

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ABSTRACT

We consider a two-dimensional nonstationary Navier–Stokes shear flow with a subdifferential boundary condition on a part of the boundary of the flow domain, namely, with a boundary driving subject to the Tresca law. There exists a unique global in time solution of the considered problem which is governed by a variational inequality. Our aim is to prove the existence of a global attractor of a finite fractional dimension and of an exponential attractor for the associated semigroup. We use the method of l-trajectories. This research is motivated by a problem from lubrication theory.

1. Introduction

Remarking on future directions of research in the field of contact mechanics, in their recent book [1], the authors wrote: "The infinite-dimensional dynamical systems approach to contact problems is virtually nonexistent. (…) This topic certainly deserves further consideration".

From the mathematical point of view a considerable difficulty in analysing problems of contact mechanics, and dynamical problems in particular, comes from the presence of involved boundary constraints which are often modelled by boundary conditions of a dissipative subdifferential type and lead to a formulation of the considered problem in terms of a variational or hemivariational inequality with, frequently, nondifferentiable boundary functionals.

Our aim in this paper is to contribute to this topic by an examination of the large time behaviour of solutions of a problem coming from the theory of lubrication.

We study the problem of existence of the global attractor of a finite fractal dimension and of an exponential attractor for a class of two-dimensional turbulent *boundary driven* flows subject to the Tresca law which naturally appears in lubrication theory. The existence of such attractors strongly suggest that the time asymptotics of the considered flow can be described by a finite number of parameters and then treated numerically [2,3]. We study the problem in its weak formulation given in terms of an evolutionary variational inequality with a nondifferentiable boundary functional. This situation produces an obstacle for applying directly the classical methods, presented e.g., in monographs [3–7], to prove that the fractal dimension of the global attractor is finite. Instead, we apply the powerful method of l-trajectories, introduced in [8,9] which we use further to prove the existence of an exponential attractor. The method of l-trajectories helps us to prove the existence of an exponential attractor for a considerably large class of nonlinear problems, in particular that with lack of good regularity properties (c.f., e.g., [10–12] and references therein).

The problem we consider is as follows. The flow of an incompressible fluid in a two-dimensional domain Ω is described by the equation of motion

$$u_t - \nu \Delta u + (u \cdot \nabla) u + \nabla p = 0 \quad \text{in } \Omega \tag{1.1}$$

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and the incompressibility condition

$$\operatorname{div} u = 0 \quad \text{in } \Omega. \tag{1.2}$$

To define the domain Ω of the flow, let Ω_∞ be the channel,

$$\Omega_\infty = \{x = (x_1, x_2) : -\infty < x_1 < \infty,\ 0 < x_2 < h(x_1)\},$$

where h is a positive function, smooth, and L-periodic in x_1. Then we set

$$\Omega = \{x = (x_1, x_2) : 0 < x_1 < L,\ 0 < x_2 < h(x_1)\}$$

and $\partial\Omega = \bar{\Gamma}_0 \cup \bar{\Gamma}_L \cup \bar{\Gamma}_1$, where Γ_0 and Γ_1 are the bottom and the top, and Γ_L is the lateral part of the boundary of Ω.

We are interested in solutions of (1.1)–(1.2) in Ω which are L-periodic with respect to x_1. We assume that

$$u = 0 \quad \text{at } \Gamma_1. \tag{1.3}$$

Moreover, we assume that there is no flux condition across Γ_0 so that the normal component of the velocity on Γ_0 satisfies

$$u \cdot n = 0 \quad \text{at } \Gamma_0, \tag{1.4}$$

and that the tangential component of the velocity u_η on Γ_0 is unknown and satisfies the Tresca friction law with a constant and positive maximal friction coefficient k. This means that, c.f., e.g., [1,13],

$$\left.\begin{array}{l} |\sigma_\eta(u,p)| \le k \\ |\sigma_\eta(u,p)| < k \Rightarrow u_\eta = U_0 e_1 \\ |\sigma_\eta(u,p)| = k \Rightarrow \exists \lambda \ge 0 \text{ such that } u_\eta = U_0 e_1 - \lambda \sigma_\eta(u,p) \end{array}\right\} \quad \text{at } \Gamma_0 \tag{1.5}$$

where σ_η is the tangential component of the stress tensor on Γ_0 and $U_0 e_1 = (U_0, 0)$, $U_0 \in \mathbb{R}$, is the velocity of the lower surface producing the driving force of the flow.

If $n = (n_1, n_2)$ is the unit outward normal to Γ_0, and $\eta = (\eta_1, \eta_2)$ is the unit tangent vector to Γ_0 then we have

$$\sigma_\eta(u,p) = \sigma(u,p) \cdot n - ((\sigma(u,p) \cdot n) \cdot n)n, \tag{1.6}$$

where $\sigma(u,p) = (\sigma_{ij}(u,p)) = (-p\delta_{ij} + \nu\left(u_{i,j} + u_{j,i}\right))$ is the stress tensor. Finally, the initial condition for the velocity field is

$$u(x,0) = u_0(x) \quad \text{for } x \in \Omega.$$

The problem is motivated by the examination of a certain two-dimensional flow in an infinite (rectified) journal bearing $\Omega \times (-\infty, +\infty)$, where $\Gamma_1 \times (-\infty, +\infty)$ represents the outer cylinder, and $\Gamma_0 \times (-\infty, +\infty)$ represents the inner, rotating cylinder. In the lubrication problems the gap h between cylinders is never constant. We can assume that the rectification does not change the equations as the gap between cylinders is very small with respect to their radii.

The knowledge or the judicious choice of the boundary conditions on the fluid–solid interface is of particular interest in lubrication area which is concerned with thin film flow behaviour. The boundary conditions to be employed are determined by numerous physical parameters characterizing, for example, surface roughness and rheological properties of the fluid.

The widely used no-slip condition when the fluid has the same velocity as surrounding solid boundary is not respected if the shear rate becomes too high (no-slip condition is induced by chemical bounds between the lubricant and the surrounding surfaces and by the action of the normal stresses, which are linked to the pressure inside the flow; on the contrary, when tangential stresses are high they can destroy the chemical bounds and induce the slip phenomenon). We can model such situation by a transposition of the well-known friction laws between two solids [1] to the fluid–solid interface.

The system of Eqs. (1.1)–(1.2) with boundary conditions: (1.3) at Γ_1 for $h = const$ and $u = const$ on Γ_0, instead of (1.4)–(1.5), was intensively studied in several contexts, some of them mentioned in the introduction of [14]. The autonomous case with $h \neq const$ and with $u = const$ on Γ_0 was considered in [15,16]. See also [17] where the case $h \neq const$, $u = U(t)e_1$ on Γ_0, was considered. The dynamical problem, important for applications, we consider in this paper has been studied earlier in [18] in the nonautonomous case for which the existence of a pullback attractor was established with the use of a method that, however, did not guarantee the finite dimensionality of the pullback attractor (or the global attractor in the reduced autonomous case).

To establish the existence of the global attractor of a finite fractal dimension we use the method of l-trajectories as presented in [9]. This method appears very useful when one deals with variational inequalities, cf., [12], as it overcomes obstacles coming from the usual methods. One needs neither compactness of the dynamics which results from the second energy inequality nor asymptotic compactness, cf., i.e., [7,17], which results from the energy equation. In the case of variational inequalities it is sometimes not possible to get the second energy inequality and the differentiability of the associated semigroup due to the presence of nondifferentiable boundary functionals. On the other hand, we do not have an energy equation to prove the asymptotic compactness.

While there are other methods to establish the existence of the global attractor where the problem of the lack of regularity appears, that, e.g., based on the notion of the Kuratowski measure of noncompactness of bounded sets, where we do not need even the continuity of the semigroup associated with a given dynamical problem, cf., e.g., [19], and also [18], where