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Hopf bifurcation and multiple periodic solutions in Lotka–Volterra systems with symmetries☆

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ABSTRACT

The purpose of this paper is to study Hopf bifurcations in a delayed Lotka–Volterra system with dihedral symmetry. By treating the response delay as bifurcation parameter and employing equivariant degree method, we obtain the existence of multiple branches of nonconstant periodic solutions through a local Hopf bifurcation around an equilibrium. We find that competing coefficients and the response delay in the system can affect the spatio-temporal patterns of bifurcating periodic solutions. According to their symmetric properties, a topological classification is given for these periodic solutions. Furthermore, an estimation is presented on minimal number of bifurcating branches. These theoretical results are helpful to better understand the complex dynamics induced by response delays and symmetries in Lotka–Volterra systems.

1. Introduction

The Lotka–Volterra systems, also known as the predator–prey systems, are frequently used to describe the dynamics of biological systems and particularly play an important role in studying population dynamics. Another reason for the importance of such systems is that they have been successfully used in economic theory, see [1] for example. In general, response delays in a ecological system can not be avoided due to the finite response speed. Therefore, Lotka–Volterra systems with delays are more realistic and useful. In this paper, we will investigate the following n interacting species competitive Lotka–Volterra system with a response delay

$$\begin{cases} \dot{x}_1(t) = x_1(t)(r_1 - a_{11}x_1(t-\tau) - \cdots - a_{1n}x_n(t-\tau)), \\ \dot{x}_2(t) = x_2(t)(r_2 - a_{21}x_1(t-\tau) - \cdots - a_{2n}x_n(t-\tau)), \\ \vdots \quad\quad \vdots \\ \dot{x}_n(t) = x_n(t)(r_n - a_{n1}x_1(t-\tau) - \cdots - a_{nn}x_n(t-\tau)), \end{cases} \tag{1.1}$$

where $x_i(t)$ and $\dot{x}_i(t)$ represent the size and the growth rate of the i-th species at the given time t respectively, $r_i > 0$ is the intrinsic birth rate of the i-th species, $\tau > 0$ represents response delay, $a_{ii} > 0$ describes the self-inhibiting nature of the i-th species, and $a_{ij} > 0, i \neq j$ is the competing coefficient which represents the effect the j-th species has upon the i-th species, $i, j = 1, 2, \ldots, n$. Because of the importance of Lotka–Volterra systems, there are many authors investigating their dynamics, see [2–11].

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In view of the nonlinearity and response delays, the system (1.1) usually exhibits complicated dynamical behavior. For example, the occurrence of a Hopf bifurcation may allow the considered species to fluctuate cyclically, see [3,7,4,5]. Moreover, it is able to establish existence of multiple periodic solutions with various symmetric properties in the system (1.1) when such a system is inherent in some symmetry (see [12,2]). Therefore, in order to better understand the complex dynamics of the system (1.1), it is of great significance to assume that the considered species are approximately identical and there is some symmetry in the system (1.1).

Without exploiting symmetries, Hirano and Rybicki [13] apply the S^1-equivariant degree theory to obtain a nonconstant periodic solution for the delayed system (1.1). Following the original idea in [13], Hirano, Krawcewicz and Ruan [12] establish the existence of multiple nonconstant periodic solutions for the system (1.1) with symmetries, and classify the obtained periodic solutions according to their symmetric properties. The approach employed in [12] is equivariant degree, which can be computerized and applied even in non-symmetric problems. In [12,13], the existence of periodic solutions is achieved by the technique of homotopy argument, and it is assumed that the linearized system has no purely imaginary eigenvalues (see the inequality (1.5) in [13] and (H3) in [12]). We should point out that the method used in [13,12] is very challenging and important to discuss the existence of periodic solutions. As is well known, the existence of purely imaginary eigenvalues may lead to complicated dynamics especially occurrence of Hopf bifurcations. The phenomena of Hopf bifurcations are discussed in [7,3–5] for Lotka–Volterra systems, but there are no symmetries to be considered in the systems and it should be assumed that the characteristic equations have only one pair of simple purely imaginary roots. Moreover, the existence of multiple nonconstant periodic solutions is not obtained in the references cited above.

Based on this motivation, regarding the response delay as a bifurcation parameter, we will discuss the existence of multiple periodic solutions for the system (1.1) through a local Hopf bifurcation around an equilibrium. In addition, we also exhibit various symmetric properties for the bifurcating nonconstant periodic solutions. For this purpose, we will use the twisted equivariant degree method presented in [14], which has following important advantages: (1) it can be applied in the general setting of autonomous constant-delayed (equivariant) differential equations, for the purpose of studying (equivariant) Hopf bifurcations; (2) a standard outcome expected from this method includes: existence, multiplicity (depends on the phase symmetry), symmetric property (using maximal isotropies) and a lower estimate on the number of branches (as a nature of topological tool, only a lower bound is possible); (3) usage of this method in applications (thanks to a computational formula of the related bifurcation invariant) usually reduces to evaluations of the spectrum of the linearized system at the equilibrium, and knowledge of basic degrees. Except for the equivariant degree method, there are some other methods and techniques to study the Hopf bifurcations for autonomous delayed differential equations (DDEs), such as the standard Hopf theory (see [15,16]), and symmetric Hopf bifurcation theory (see [17,18]). We remark that in fact the presence of symmetries often causes the purely imaginary eigenvalues to be multiple. Hence generally the standard Hopf bifurcation theory cannot be applied in a symmetric system. In recent years, the method presented in [17] takes an important position in the study of equivariant Hopf bifurcation, but this method often requires a non-resonance condition. Actually, the condition is not necessary for the existence of bifurcation of periodic solutions.

In order to avoid unnecessary complexity, we restrict (1.1) to the following three dimensional symmetric competitive coupled systems

$$\dot{x}_i(t) = x_i(t)[r - cx_i(t-\tau) - dx_{i-1}(t-\tau) - dx_{i+1}(t-\tau)], \quad i(\text{mod } 3), \tag{1.2}$$

where r, c, d are positive constants. There are several reasons why we are particularly interested in such a system as (1.2). First of all, the Lotka–Volterra system described by (1.2) is of minimal size for the system (1.1) with such a dihedral symmetry and can be found in a variety of population structure. Secondly, much progress has been made in the theory of dynamics (in particular for the local Hopf bifurcations) of non-symmetric Lotka–Volterra systems for one or two interacting species, and it is natural to see how these results can be extended to symmetric Lotka–Volterra systems. Thirdly, although the system (1.2) is a little simple, it allows us to have an in-depth analysis and to gain insight into possible mechanisms behind the observed behaviors. Finally, we think that the method for studying the system (1.2) could be similarly used for the system (1.1) with D_n-symmetry for large n, but there may be some difficulties in computing the twisted equivariant degree, since the lattices of orbit types rely on the number n.

The rest of this paper is organized as follows: In Section 2, we provide a short explanation of the twisted equivariant degree and its properties. Section 3 is devoted to the description of how to apply the twisted equivaiant degree method to study the occurrence of Hopf bifurcations in parameterized autonomous DDEs with symmetries. In Section 4, we investigate the symmetric Hopf bifurcation for the system (1.2) and exhibit various symmetric properties for the bifurcating periodic solutions. The paper ends with some discussions in Section 5.

2. Twisted equivariant degree

In this section, we give some basic facts on the twisted equivariant degree, which is used to investigate symmetric Hopf bifurcation for the system (1.2). For more details, we refer to [14,19].

Let Γ be a finite group and $G := \Gamma \times S^1$, where $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. Assume that W is a isometric Hilbert G-representation, namely, W is a Hilbert G-space and the inner product $\langle \cdot, \cdot \rangle$ is G-invariant, i.e., $\langle gv, gw \rangle = \langle v, w \rangle$ for all $g \in G$, $v, w \in W$. For any $x \in W$, the closed subgroup $G_x = \{g \in G : gx = x\}$ of G is called the *isotropy group* of x and the invariant subspace $G(x) := \{gx : g \in G\}$ of W is called the *orbit* of x. For a closed subgroup H of G, $N(H) = \{g : gHg^{-1} = H\}$