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A remark on global bifurcations of solutions of Ginzburg–Landau equation

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ABSTRACT

We have proved that all the closed connected sets of solutions of the complex Ginzburg–Landau equation

$$\begin{cases} -\Delta u(x) + 2i\langle A(x), \nabla u(x)\rangle + \|A(x)\|^2 u(x) = \lambda(1-|u(x)|^2)u(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

bifurcating from the set of normal solutions $\{0\} \times (0, +\infty) \subset H_0^1(\Omega, \mathbb{C}) \times (0, +\infty)$ are unbounded, where $\Omega \subset \mathbb{R}^2$ is an open, bounded domain with smooth boundary, $A(x_1, x_2) = (-x_2, x_1)$ and $\|\cdot\|$ is the usual norm in $\mathbb{R}^2$.

1. Introduction

The goal of this paper is to study nonzero solutions of the following complex Ginzburg–Landau equation

$$\begin{cases} -\Delta u(x) + 2i\langle A(x), \nabla u(x)\rangle + \|A(x)\|^2 u(x) = \lambda(1-|u(x)|^2)u(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\lambda \in (0, +\infty)$, $\Omega \subset \mathbb{R}^2$ is an open, bounded domain with smooth boundary, $A(x_1, x_2) = (-x_2, x_1)$ and $\|\cdot\|$ is the usual norm in $\mathbb{R}^2$.

Starting from one normal solution, a natural way of finding new solutions is to increase the parameter λ from 0 and to look for bifurcation values of λ. Bifurcations of solutions of the Ginzburg–Landau type problems have been considered by many authors; see for instance [1–8] and references therein. The only possible bifurcation points of solutions of problem (1.1) are the eigenvalues of the magnetic Laplacian. Usually the authors study local bifurcations of nonzero solutions of problem (1.1) by using the Crandall–Rabinowitz bifurcation theorem, Krasnosiel'ski bifurcation theorem for potential operators, Lyapunov–Schmidt reduction, center manifold theorem, theorem on attractor bifurcations and implicit function theorem. On the other hand, the global bifurcations of solutions of the one-dimensional Ginzburg–Landau model have been studied in [3]. Using the Brouwer degree the authors have proved the existence of a closed connected set of asymmetric solutions which connect the global curve of symmetric solutions to an asymmetric normal state solution.

We proceed in this paper another approach. Namely, we investigate global bifurcations in the sense of Rabinowitz, see [9], of solutions of the complex Ginzburg–Landau equation

$$\begin{cases} -\Delta u(x) + 2i\langle A(x), \nabla u(x)\rangle + \|A(x)\|^2 u(x) = \lambda(1-|u(x)|^2)u(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

bifurcating from the set of normal solutions $\{0\} \times \mathbb{R} \subset H_0^1(\Omega, \mathbb{C}) \times \mathbb{R}$.

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It is easy to verify that if $(u,\lambda)\in H_0^1(\Omega,\mathbb{C})\times\mathbb{R}$ is a solution of problem (1.2) then so is $(e^{i\theta}u,\lambda)\in H_0^1(\Omega,\mathbb{C})\times\mathbb{R}$ for every $\theta\in\mathbb{R}$. Therefore we consider solutions of problem (1.2) as critical $SO(2)$-orbits of a family of $SO(2)$-invariant functionals $J\in C^2(H_0^1(\Omega,\mathbb{R})\times H_0^1(\Omega,\mathbb{R})\times\mathbb{R},\mathbb{R})$ defined by (2.4).
Since the gradient $\nabla_u J\in C^1(H_0^1(\Omega,\mathbb{R})\times H_0^1(\Omega,\mathbb{R})\times\mathbb{R},H_0^1(\Omega,\mathbb{R})\times H_0^1(\Omega,\mathbb{R}))$ is $SO(2)$-equivariant, in order to study solutions of the equation $\nabla_u J(u,\lambda)=0$ we apply a version of the famous Rabinowitz alternative for critical $SO(2)$-orbits of $SO(2)$-equivariant gradient operators, see Theorems 4.1, 4.7 of [10]. It is worth pointing out that to prove this theorem we have used the degree theory for equivariant gradient maps, see [11–13]. In fact we have excluded one of possibilities of behavior of every continuum (closed connected set) of nonzero solutions of problem (1.2) bifurcating from the set of normal solutions. Namely, we have proved that these continua are unbounded.

2. Preliminaries

Set $S^1=\{z\in\mathbb{C}:|z|=1\}$, denote by $SO(2)$ the group of special orthogonal maps of $\mathbb{R}^2$ i.e. $SO(2)=\left\{g(\theta)=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}:\theta\in\mathbb{R}\right\}$ and consider $\mathbb{R}^2$ as $SO(2)$-representation with the $SO(2)$-action given by $(g(\theta),x)=g(\theta)x$.
Let $\Omega\subset\mathbb{R}^2$ be an open, bounded domain with smooth boundary. Consider a Sobolev spaces $\mathbb{H}=H_0^1(\Omega,\mathbb{R})\oplus H_0^1(\Omega,\mathbb{R})$ and $H_0^1(\Omega,\mathbb{C})$ with the following scalar products

$$\langle(v_1,z_1),(v_2,z_2)\rangle_{\mathbb{H}}=\int_\Omega(\nabla v_1(x),\nabla v_2(x))+v_1(x)v_2(x)\mathrm{d}x+\int_\Omega(\nabla z_1(x),\nabla z_2(x))+z_1(x)z_2(x)\mathrm{d}x$$

and $\langle u_1,u_2\rangle_{H_0^1(\Omega,\mathbb{C})}=\int_\Omega(\nabla u_1(x),\overline{\nabla u_2(x)})+u_1(x)\overline{u_2(x)}\mathrm{d}x$, where $(\cdot,\cdot)$ is the usual scalar product in $\mathbb{R}^2$. It is clear that $\langle(v_1,z_1),(v_2,z_2)\rangle_{\mathbb{H}}=\Re\langle v_1+\mathrm{i}z_1,v_2+\mathrm{i}z_2\rangle_{H_0^1(\Omega,\mathbb{C})}$.

Let $A:\mathbb{R}^2\to\mathbb{R}^2$ be a linear map given by $A(x_1,x_2)=(-x_2,x_1)$. It is clear that $A(g(\theta)x)=g(\theta)A(x)$ for every $\theta\in\mathbb{R}$.

Define a scalar product $\langle\cdot,\cdot\rangle_{H_A^1}:H_0^1(\Omega,\mathbb{C})\oplus H_0^1(\Omega,\mathbb{C})\to\mathbb{C}$ as follows

$$\langle u_1,u_2\rangle_{H_A^1}=\int_\Omega\langle\nabla u_1(x)-\mathrm{i}u_1(x)A(x),\overline{\nabla u_2(x)-\mathrm{i}u_2(x)A(x)}\rangle\mathrm{d}x.$$

We underline that norms $\|\cdot\|_{H_A^1},\|\cdot\|_{H_0^1(\Omega,\mathbb{C})}:H_0^1(\Omega,\mathbb{C})\to\mathbb{R}_+$ are equivalent, see [8]. Define a scalar product $(\cdot,\cdot)_A:\mathbb{H}\oplus\mathbb{H}\to\mathbb{R}$ by $((v_1,z_1),(v_2,z_2))_A=\Re\langle v_1+\mathrm{i}z_1.v_2+\mathrm{i}z_2\rangle_{H_A^1}$. Put $u=v+\mathrm{i}z\in H_0^1(\Omega,\mathbb{C})$. Since $\|v+\mathrm{i}z\|_{H_0^1(\Omega,\mathbb{C})}=\|(v,z)\|_{\mathbb{H}}$ and $\|v+\mathrm{i}z\|_{H_A^1}=\|(v,z)\|_A$, the norms $\|\cdot\|_A,\|\cdot\|_{\mathbb{H}}:\mathbb{H}\to\mathbb{R}_+$ are equivalent.

Remark 2.1. The Hilbert spaces $(\mathbb{H},\langle\cdot,\cdot\rangle_{\mathbb{H}})$, $(\mathbb{H},\langle\cdot,\cdot\rangle_A)$ are orthogonal $SO(2)$-representations with the $SO(2)$-action given by $(g(\theta)(v,z))(x)=(g(\theta)(v(x),z(x))^t)^t$. The Hilbert spaces $(H_0^1(\Omega,\mathbb{C}),\langle\cdot,\cdot\rangle_{H_0^1(\Omega,\mathbb{C})})$, $(H_0^1(\Omega,\mathbb{C}),\langle\cdot,\cdot\rangle_{H_A^1})$, are orthogonal S^1-representations with the S^1-action defined by $(e^{i\theta_1}u)(x)=e^{i\theta_1}u(x)$.

Define the Ginzburg–Landau potential $I\in C^2(H_0^1(\Omega,\mathbb{C})\times\mathbb{R},\mathbb{R})$ as follows

$$I(u,\lambda)=\int_\Omega\frac{1}{2}|(\nabla-\mathrm{i}A(x))u(x)|^2+\frac{\lambda}{4}(1-|u(x)|^2)^2\mathrm{d}x. \tag{2.1}$$

In other words we obtain $I(u,\lambda)=\frac{1}{2}\|u\|^2_{H_A^1}+\frac{\lambda}{4}\int_\Omega(1-|u(x)|^2)^2\mathrm{d}x$.

We are going to study solutions of the following problem

$$\nabla_u I(u,\lambda)=0. \tag{2.2}$$

The corresponding Euler–Lagrange equation has the following form

$$\begin{cases}-\Delta u(x)+2\mathrm{i}\langle A(x),\nabla u(x)\rangle+\|A(x)\|^2u(x)=\lambda(1-|u(x)|^2)u(x) & \text{in }\Omega,\\ u=0 & \text{on }\partial\Omega.\end{cases} \tag{2.3}$$

Since in this article we are going to apply abstract results of real equivariant nonlinear analysis, we replace the Ginzburg–Landau functional with the functional $J\in C^2(\mathbb{H}\times\mathbb{R},\mathbb{R})$ defined as follows

$$J((v,z),\lambda):=I(v+\mathrm{i}z,\lambda). \tag{2.4}$$

In other words $J((v,z),\lambda)=\frac{1}{2}\|v+\mathrm{i}z\|^2_{H_A^1}+\frac{\lambda}{4}\int_\Omega(1-v(x)^2-z(x)^2)^2\mathrm{d}x$.

The corresponding Euler–Lagrange system is the following

$$\begin{cases}-\Delta v(x)-2\langle A(x),\nabla z(x)\rangle+\|A(x)\|^2v(x)=\lambda(1-v(x)^2-z(x)^2)v(x) & \text{on }\Omega,\\ -\Delta z(x)+2\langle A(x),\nabla v(x)\rangle+\|A(x)\|^2z(x)=\lambda(1-v(x)^2-z(x)^2)z(x) & \text{on }\Omega,\\ v=z=0 & \text{in }\partial\Omega.\end{cases} \tag{2.5}$$

Lemma 2.1. *Functional J given by (2.4) is $SO(2)$-invariant i.e. $J(g(\theta)(v,z),\lambda)=J((v,z),\lambda)$ for every $((v,z),\lambda)\in\mathbb{H}\times\mathbb{R}$ and $\theta\in\mathbb{R}$.*