Nonlinear Analysis: Real World Applications 12 (2011) 3306–3314

Contents lists available at SciVerse ScienceDirect

Nonlinear Analysis: Real World Applications

journal homepage: www.elsevier.com/locate/nonrwa

Second order periodic problems in the presence of dry friction

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ARTICLE INFO

Article history:
Received 10 November 2009
Accepted 26 May 2011

Keywords:
Dry friction
Differential inclusion
Eigenvalues
Periodic solutions
Topological degree

ABSTRACT

We prove, via an approach by ordinary differential equations, the existence of oscillations for second order differential inclusions of the form

$$u'' + u \in \varphi(t) - \mu(u)S(u'),$$

where φ is 2π-periodic, μ is allowed to satisfy the at most linear growth condition of the form $\mu_0 \le \mu(u) \le \mu_0 + \mu_1|u|$ with some restrictions on μ_1, S is bounded and continuous in $\mathbb{R}\setminus\{0\}$ with a jump discontinuity at 0 and $S(0^-) < S(0^+)$. An existence result for resonance at first nonzero eigenvalue is obtained.

1. Introduction

Imagine a mass attached to a spring, moving in a tube containing a fluid, with contact to the wall and periodically excited by a force $\Phi(t)$. Under the simplest assumptions about the forces involved, we have $-ku$ for the spring, $-ru'$ for the viscous damping caused by the fluid and $-c\ \mathrm{sgn}\, u'$ for the dry friction (or Coulomb friction) at the wall, with positive constants k, r and c. Hence, balance of forces and appropriate scaling of time t yield the engineering standard form

$$u'' + 2Du' + \mu \mathrm{sgn}\, u' + u = \varphi(t), \tag{1.1}$$

where $2D = r/\sqrt{Mk}$, $\mu = k/c$ and $\varphi(t) = \Phi(\nu t)/k$ with $\nu = \sqrt{M/k}$. We note that in more realistic cases μ may depend on the position of the mass (see e.g. [1]), where dry friction often leads to

$$\mu(r) = \mu_0 + \mu_1|r|, \tag{1.2}$$

and some of these forces may actually be nonlinear (see e.g. Section 76 of [2] or Section 50 of [3]). Thus, we need to consider

$$u'' + g(u') + \mu(u)S(u') + f(u) = \varphi(t), \tag{1.3}$$

where f, g are continuous, φ is periodic, and S is bounded and continuous in $\mathbb{R}\setminus\{0\}$ with a jump discontinuity at 0 and $S(0^-) < S(0^+)$. We refer the reader to the survey by Kunze [4] for a wealth of information on this kind of problem.

The case when $f(u) = u,\ g(u') = 0,\ \mu(u) \equiv \mu$ and $S(u') = \mathrm{sgn}\, u'$ has been considered by Deimling and Szilágyi [5], Deimling [6] and Cabada and Sanchez [7]. Existence results of (1.3) for resonance at λ_2 were obtained in these papers, where λ_2 is the first nonzero eigenvalue of

$$u''(t) + \lambda u(t) = 0,$$
$$u(0) = u(2\pi), \qquad u'(0) = u'(2\pi)$$

and $\lambda_2 = 1$. However $\mu(u)$ in [5,6] is a constant and $\mu(u)$ in [7] is bounded in $\mathbb{R}$.

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doi:10.1016/j.nonrwa.2011.05.027

The periodic problem (1.3) has also been considered by Deimling [8], and Bothe [9], and existence results for resonance at the zero eigenvalue ($\lambda_1 = 0$) were obtained in these two papers. In [8], $\mu(u)$ is assumed to be a constant; while in [9], $\mu(u)$ is allowed to satisfy a condition like (1.2). Notice that the eigenspace corresponding to λ_1 $(= 0)$ is span$\{c\}$, and the eigenspace corresponding to λ_2 $(= 1)$ is span$\{\frac{1}{\sqrt{\pi}} \cos t, \frac{1}{\sqrt{\pi}} \sin t\}$. Since they are of different dimension, there exists a large difference in the processes of proving existence results for resonances at λ_1 and λ_2.

So, the natural question is whether or not the existence results for resonance at the first nonzero eigenvalue could be established under unbounded $\mu(u)$.

It is the purpose of this paper to prove the existence of solutions of

$$\begin{aligned} &u'' + u + \mu(u)S(u') = \varphi(t), \\ &u(0) = u(2\pi), \qquad u'(0) = u'(2\pi) \end{aligned} \tag{1.4}$$

for resonance to $\lambda_2 = 1$ when $\mu(x)$ is at most linear growth, i.e., $\mu_0 \le \mu(u) \le \mu_0 + \mu_1|u|$ with some restrictions on μ_1, see (H3). We make the following assumptions.

(H1) S is a function defined, bounded and continuous in $\mathbb{R} \setminus \{0\}$ with a jump discontinuity at 0, $S(0^-) < S(0^+)$, and

$$\alpha := \limsup_{z\to-\infty} \mathcal{S}(z) < 0 < \beta := \liminf_{z\to+\infty} \mathcal{S}(z).$$

(H2) φ is continuous, 2π-periodic.

(H3) $\mu : \mathbb{R} \to \mathbb{R}$ is continuous and there exist $\mu_0 \in (0, \infty)$, $\mu_1 \in [0, \infty)$, such that

$$\mu_0 \le \mu(u) \le \mu_0 + \mu_1|u|, \quad u \in \mathbb{R}.$$

Remark 1.0. Notice that in the mechanical system studied in [1], friction leads to $\mu(u) = \mu_0 + \mu_1|u|$ with $\mu_0, \mu_1 > 0$.

Since S is not properly defined for $z = 0$, (1.4) is understood as

$$\begin{aligned} &u'' + u \in \varphi(t) - \mu(u)\mathcal{S}(u'), \\ &u(0) = u(2\pi), \qquad u'(0) = u'(2\pi) \end{aligned} \tag{1.5}$$

with

$$\mathcal{S}(z) = \begin{cases} S(z), & \text{for } z \neq 0, \\ [S(0^-), S(0^+)], & \text{for } z = 0. \end{cases} \tag{1.6}$$

Definition. By a *solution* of (1.4) we mean a 2π-periodic function $u \in W^{2,2}(0, 2\pi)$ such that there exists $w \in L^\infty(0, 2\pi)$ satisfying $S(0^-) \le w(t) \le S(0^+)$ a.e. in $B := \{t : u'(t) = 0\}$, $w(t) = S(u'(t))$ a.e. in $A := \{t : u'(t) \neq 0\}$ and

$$u''(t) + \mu(u(t))w(t) + u(t) = \varphi(t), \quad t \in (0, 2\pi),$$

$$u(0) = u(2\pi), \qquad u'(0) = u'(2\pi).$$

Let E be the Banach space $L^2(0, 2\pi)$ with the inner product

$$\langle u, v\rangle = \int_0^{2\pi} u(t)v(t)\mathrm{d}t.$$

Since (1.5) contains in the left-hand side a non-invertible operator, we shall use the decomposition of functional space into its kernel and a complementary subspace. Let us set

$$E = E_1 \oplus E_2,$$

where $\oplus$ denotes orthogonal direct sum,

$$E_1 := \text{span}\{\varphi_1(t),\ \varphi_2(t)\}$$

and

$$\varphi_1(t) = \frac{1}{\sqrt{\pi}} \cos t, \qquad \varphi_2(t) = \frac{1}{\sqrt{\pi}} \sin t.$$

Accordingly, we split each $u \in L^2(0, 2\pi)$ as $u = u_1 + u_2$, $u_i \in E_i$, $i = 1, 2$. Set

$$S^* := \sup_{\mathbb{R}\setminus\{0\}} |S(z)|. \tag{1.7}$$