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History-dependent subdifferential inclusions and hemivariational inequalities in contact mechanics[☆]

Stanisław Migórski[a,*], Anna Ochal[a], Mircea Sofonea[b]

[a] Faculty of Mathematics and Computer Science, Jagiellonian University, Institute of Computer Science, ul. Stanisława Łojasiewicza 6, 30348 Krakow, Poland

[b] Laboratoire de Mathématiques et Physique, Université de Perpignan, 52 Avenue Paul Alduy, 66860 Perpignan, France

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ABSTRACT

We consider a class of subdifferential inclusions involving a history-dependent term for which we provide an existence and uniqueness result. The proof is based on arguments on pseudomonotone operators and fixed point. Then we specialize this result in the study of a class of history-dependent hemivariational inequalities. Such kind of problems arises in a large number of mathematical models which describe quasistatic processes of contact between a deformable body and an obstacle, the so-called foundation. To provide an example we consider a viscoelastic problem in which the frictional contact is modeled with subdifferential boundary conditions. We prove that this problem leads to a history-dependent hemivariational inequality in which the unknown is the velocity field. Then we apply our abstract result in order to prove the unique weak solvability of the corresponding contact problem.

1. Introduction

Nonlinear inclusions and hemivariational inequalities play an important role in the study of both the qualitative and numerical analysis of nonlinear boundary value problems arising in Mechanics, Physics and Engineering Science. For this reason, the mathematical literature dedicated to this field is extensive and the progress made in the past two decades is impressive. At the heart of this theory is the intrinsic inclusion of free boundaries in an elegant mathematical formulation. The analysis of nonlinear inclusions and hemivariational inequalities, including existence and uniqueness results, can be found in [1–3].

Phenomena of contact between deformable bodies abound in industry and everyday life. Contact of braking pads with wheels, tires with roads, pistons with skirts are just a few simple examples. Common industrial processes such as metal forming and metal extrusion involve contact evolutions. Owing to their inherent complexity, contact phenomena lead to mathematical models expressed in terms of strongly nonlinear elliptic or evolutionary boundary value problems.

Considerable progress has been achieved recently in modeling, mathematical analysis and numerical simulations of various contact processes and, as a result, a general Mathematical Theory of Contact Mechanics is currently emerging. It is concerned with the mathematical structures which underlie general contact problems with different constitutive laws, i.e. materials, varied geometries and different contact conditions. Its aim is to provide a sound, clear and rigorous background to the constructions of models for contact, proving existence, uniqueness and regularity results, assigning

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* Corresponding author.
E-mail address: Migorski@ii.uj.edu.pl (S. Migórski).

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precise meaning to solutions, among others. To this end, it operates with various mathematical concepts which include variational and hemivariational inequalities and multivalued inclusions, as well. The variational analysis of various contact problems, including existence and uniqueness results, can be found in the monographs [4–10]. The state of the art in the field can also be found in the proceedings [11–13] and in the special issue [14], as well.

The aim of this paper is to prove new existence and uniqueness results in the study of a class of subdifferential inclusions and hemivariational inequalities, and to apply these results in the analysis of a quasistatic contact problem. The trait of novelty of our work arise in the special structure of the abstract problems we consider, which are governed by an operator depending on the history of the solution. The class of hemivariational inequalities we study represents a general framework in which a large number of quasistatic contact problems, associated with various constitutive laws and frictional contact conditions, can be cast. Therefore, the paper provides arguments and tools which can be useful to prove the unique solvability of a large number of quasistatic contact problems.

The rest of the paper is structured as follows. In Section 2, we prove the existence and uniqueness of the solution to a class of history-dependent subdifferential inclusions. In Section 3, we specialize this result in the study of history-dependent hemivariational inequalities. In Section 4, we introduce a quasistatic frictional contact problem in which the material's behavior is modeled with a viscoelastic constitutive law and the frictional contact conditions are in a subdifferential form. We show that this problem leads to a history-dependent hemivariational inequality for the velocity field. Then, we use our theoretic results in the analysis of this contact problem and prove its unique solvability.

2. History-dependent subdifferential inclusions

Let $\Omega \subset \mathbb{R}^d$ be an open bounded subset of $\mathbb{R}^d$ with a Lipschitz continuous boundary $\partial\Omega$ and $\Gamma \subseteq \partial\Omega$. Let V be a closed subspace of $H^1(\Omega;\mathbb{R}^s)$, $s \geq 1$, $H = L^2(\Omega;\mathbb{R}^s)$ and $Z = H^{1/2}(\Omega;\mathbb{R}^s)$. Denoting by $i: V \to Z$ the embedding, by $\gamma: Z \to L^2(\Gamma;\mathbb{R}^s)$ and $\gamma_0: H^1(\Omega;\mathbb{R}^s) \to H^{1/2}(\Gamma;\mathbb{R}^s) \subset L^2(\Gamma;\mathbb{R}^s)$ the trace operators, we get $\gamma_0 v = \gamma(iv)$ for all $v \in V$. For simplicity, in what follows, we omit the notation of the embedding i and we write $\gamma_0 v = \gamma v$ for all $v \in V$. Given a normed space X, we denote by 2^X the collection of all subsets of X and by $\langle \cdot,\cdot \rangle_{X^*\times X}$ the duality pairing between X and its dual X^*. The symbol w-X is used for the space X endowed with the weak topology. The space of all linear and continuous operators from a normed space X to a normed space Y is denoted by $\mathcal{L}(X, Y)$.

From the theory of Sobolev spaces, we know that (V, H, V^*) and (Z, H, Z^*) form evolution triples of spaces and the embedding $V \subset Z$ is compact. We denote by c_e the embedding constant of V into Z, by $\|\gamma\|$ the norm of the trace in $\mathcal{L}(Z, L^2(\Gamma;\mathbb{R}^s))$ and by $\gamma^*: L^2(\Gamma;\mathbb{R}^s) \to Z^*$ the adjoint operator to γ. We also introduce the following spaces

$$\mathcal{V} = L^2(0,T;V), \qquad \mathcal{Z} = L^2(0,T;Z) \quad \text{and} \quad \widehat{\mathcal{H}} = L^2(0,T;H),$$

where $0 < T < +\infty$. Since the embeddings $V \subseteq Z \subseteq H \subseteq Z^* \subseteq V^*$ are continuous, it is known that the embeddings $\mathcal{V} \subseteq \mathcal{Z} \subseteq \widehat{\mathcal{H}} \subseteq \mathcal{Z}^* \subseteq \mathcal{V}^*$ are also continuous, where $\mathcal{Z}^* = L^2(0,T;Z^*)$ and $\mathcal{V}^* = L^2(0,T;V^*)$.

Let $A: (0,T) \times V \to V^*$, $\mathcal{S}: \mathcal{V} \to \mathcal{V}^*$, $J: (0,T) \times L^2(\Gamma;\mathbb{R}^s) \to \mathbb{R}$ and $f: (0,T) \to V^*$ be given. Then, in this section, we consider the following subdifferential problem.

Problem 1. Find $u \in \mathcal{V}$ such that

$$A(t, u(t)) + \mathcal{S}u(t) + \gamma^* \partial J(t, \gamma u(t)) \ni f(t) \quad \text{a.e. } t \in (0,T). \tag{1}$$

To avoid any confusion, we note that in (1) and below the notation $\mathcal{S}u(t)$ stands for $(\mathcal{S}u)(t)$, i.e. $\mathcal{S}u(t) = (\mathcal{S}u)(t)$ for all $u \in \mathcal{V}$ and a.e. $t \in (0,T)$. The symbol ∂J denotes the Clarke subdifferential of a locally Lipschitz function $J(t,\cdot)$ (cf. Chapter 2.1 of [1]).

We complete the statement of Problem 1 with the following definition.

Definition 2. A function $u \in \mathcal{V}$ is called a solution to Problem 1 if and only if there exists $\zeta \in \mathcal{Z}^*$ such that

$$\begin{cases} A(t, u(t)) + \mathcal{S}u(t) + \zeta(t) = f(t) \quad \text{a.e. } t \in (0,T), \\ \zeta(t) \in \gamma^* \partial J(t, \gamma u(t)) \quad \text{a.e. } t \in (0,T). \end{cases}$$

In order to provide the solvability of Problem 1, we need the following hypotheses.

$$\left.\begin{array}{l} A: (0,T) \times V \to V^* \text{ is such that} \\ \text{(a) } A(\cdot, v) \text{ is measurable on } (0,T) \text{ for all } v \in V; \\ \text{(b) } A(t,\cdot) \text{ is hemicontinuous and strongly monotone for a.e.} \\ \quad t \in (0,T), \text{ i.e. } \langle A(t,v_1) - A(t,v_2), v_1 - v_2 \rangle_{V^*\times V} \\ \quad \geq m_1 \|v_1 - v_2\|_V^2 \text{ for all } v_1, v_2 \in V \text{ with } m_1 > 0; \\ \text{(c) } \|A(t,v)\|_{V^*} \leq a_0(t) + a_1\|v\|_V \text{ for all } v \in V, \text{ a.e. } t \in (0,T) \\ \quad \text{with } a_0 \in L^2(0,T),\ a_0 \geq 0 \text{ and } a_1 > 0; \\ \text{(d) } A(t,0) = 0 \text{ for a.e. } t \in (0,T). \end{array}\right\} \tag{2}$$